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This curriculum bulletin is one of a planned series of bulletins designed to meet the needs of teachers and supervisors. The materials in this bulletin consist of a series of daily lesson plans for use by teachers in presenting a modern program of seventh year mathematics. In these lesson plans are developed the concepts, skills, and applications of "Mathematics Seventh Year." The material in this bulletin emphasizes (1) an understanding of mathematical structure, (2) growth of a number system, (3) relations and operations in a number system, (4) a development of mathematical skills based on an understanding of mathematical principles, and (5) concept of set in number and in geometry. The bulletin is organized into five chapters: Numbers and Numerals, Operations and Properties of Whole Numbers, Non-Metric Geometry, Factoring, and Rational Numbers (Multiplication and Division). (RP)

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# MATHEMATICS

## 7th YEAR

## PART 1

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*Board of Education • City of New York*

SE005 622

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CURRICULUM BULLETIN • 1966-67 SERIES • No. 3a

# MATHEMATICS

## 7th YEAR

## PART 1

BOARD OF EDUCATION OF THE CITY OF NEW YORK

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# FOREWORD

At a time when our society is increasingly dependent upon mathematically literate citizens and upon trained mathematical manpower, it is essential that vital and contemporary mathematics be taught in our schools.

The mathematics program set forth in this publication has developed as a result of experimentation and evaluation in classroom situations. This is Part I of Mathematics Seventh Year. Part II, a separate bulletin, will be published during the school year 1966-67.

This bulletin represents a cooperative effort of the Office of Junior High Schools, the Bureau of Mathematics and the Bureau of Curriculum Development.

We wish to thank the staff members who have so generously contributed to this work.

# CONTENTS

Page

Chapter I NUMBERS AND NUMERALS 1

Sets; one-to-one correspondence; sets and numbers; equivalent and equal sets; Roman numeration system; Hindu-Arabic notation; base ten; expanded numeral; exponents; place value; number bases; base five.

Chapter II OPERATIONS AND PROPERTIES OF WHOLE NUMBERS 39

Sets and addition; union of sets; properties of addition; subtraction; proof; problem-solving involving addition and subtraction; multiplication and sets; properties of multiplication; distributive property; short form of division; divisors greater than 99; problem solving involving multiplication and division.

Chapter III NON-METRIC GEOMETRY 97

Points and space; line segments and lines; rays and angles; planes; planes and lines; simple closed curves; polygons; parallel lines; space figures.

Chapter IV FACTORING 139

Intersection of sets; Venn diagrams; factors; divisibility; prime numbers; composite numbers; unique factorization property; greatest common factor.

Chapter V RATIONAL NUMBERS (Multiplication and Division) 162

Meaning of rational number; fractions and rational numbers; multiplication of rational numbers; properties of multiplication; simplest form; least common multiple; comparing rational numbers; rational numbers and the number line; density property; mixed fractional form; reciprocal (multiplicative inverse); division by one; rational numbers in division.

## INTRODUCTION

The materials in this bulletin consist of a series of daily lesson plans for use by teachers in presenting a modern program of seventh year mathematics. In these lesson plans are developed the concepts, skills, and applications of Mathematics Seventh Year. There is an emphasis on:

an understanding of mathematical structure
growth of a number system
relations and operations in a number system
a development of mathematical skills based on an understanding of mathematical principles
concept of set in number and in geometry

This bulletin is the culmination of several years of experimentation involving the cooperative efforts of the Division of Curriculum Development and the Junior High School Division. The mathematics presented in this bulletin is based upon concepts and skills which were developed in previous grades. The eighth and ninth year mathematics courses will extend the basic ideas of Mathematics Seventh Year.

The materials in this program have been tried out over a period of years in schools in all five boroughs of the city. The materials in the lesson plans reflect the classroom tryout and continued evaluation by teachers and supervisors. They have been revised a number of times in the light of these evaluations.

## ORGANIZATION OF THIS BULLETIN

The content of this bulletin is arranged in the sequence in which it is to be used. It is expected that a lesson will be presented before the next numbered lesson and that each chapter will be presented before any work in the ensuing chapter is begun. Although this may seem to be a departure from the cyclical arrangement of materials found in earlier curriculum bulletins on seventh year mathematics, a cyclical approach is in fact an integral part of each chapter. For example, understanding of the concept of a number system is developed on progressively higher levels as pupils advance from an understanding of whole numbers to rational numbers to signed numbers.

Various topics for enrichment have been included. Labeled optional, they have been placed with the topics of which they are a logical outgrowth.

## SUGGESTED PROCEDURE FOR USING THIS BULLETIN

It is suggested that the following procedure be considered in using this publication:

Read the entire bulletin before making plans to teach any part of it. Read each chapter in turn to become acquainted with the content and spirit of Mathematics Seventh Year and with the relationships among the topics in the course.

Study the introductory discussion in each chapter you plan to present. Note the relationship of each lesson to the one preceding it and the one following it. Each lesson is organized in terms of:

1. Topic
2. Aim
3. Specific Objectives
4. Procedure
5. Practice
6. Summary Questions

Amplify the practice material suggested for each lesson with additional material from suitable textbooks.

Practice in computation and in the solution of verbal problems should not be confined to the sections in which this work appears in the bulletin, but should be interspersed among other topics in order to sustain interest and provide for continuous development and reinforcement of computational skills and of problem-solving skills.

## EVALUATION

An evaluation program includes not only the checking of completed work at convenient intervals, but also continued appraisal. It is a general principle of evaluation that results are checked against objectives. The objectives of this course include concepts, principles and understandings, as well as skills.

Written tests are the most frequently used instrument for evaluation and remain the chief rating tool of the teacher. Test items should be designed to test not only recall of factual items, but also the ability of the pupil to make intelligent application of mathematical principles. Some of the writing activities which teachers may use for the purpose of evaluation include:

written tests
written homework assignments
keeping of notebooks
special reports
quizzes

To evaluate pupil understanding continually, there are a number of oral activities which teachers may use such as:

- pupil explanations of approaches used in new situations
- pupil justification of statements
- pupil restatement of problems
- pupil explanation of interrelationship of ideas
- pupil discovery of patterns
- oral quizzes
- pupil reports

Evaluation procedures also include teacher observation of pupil's work at chalkboard and of pupil's work at seat.

Self-evaluation by pupils can be encouraged through short self-marking quizzes.

## DEVELOPMENT OF THE MATHEMATICS GRADE 7 PROGRAM

During the school year 1963-1964, a revised Seventh Year Mathematics scope and sequence was developed by staff members from the Division of Curriculum Development and the Junior High School Division. This scope and sequence was the basic document for writing teams which consisted of junior high school mathematics coordinators.

Preliminary materials were prepared by these teams and were reviewed by the Junior High School Mathematics Curriculum Committee. Revisions were made on the basis of the Committee's suggestions. In September 1964, the first draft of the materials was ready and was made available to teachers who were to take part in their experimental use.

These preliminary materials were tried out on an experimental basis for the first time in selected junior high schools during the school year 1964-65. A program of evaluation of these materials was set up which included chapter by chapter evaluation reports from classroom teachers, junior high school coordinators, and supervisors of mathematics in pilot schools. The materials were then revised accordingly.

The school year 1965-1966 saw the second year of experimental use of the materials with additional schools participating. Similar evaluation procedures were followed.

Final work on Part I of this bulletin, preparing it for publication, was completed in July, 1966. It is expected that a final revision of Part II will be completed during the Fall of 1966. This revision will reflect the second year of classroom tryout.

## ACKNOWLEDGEMENTS

The preparation of this bulletin was under the general direction of Martha R. Finkler, Acting Associate Superintendent, Junior High School Division, Margaret Bible, Acting Assistant Superintendent, Junior High School Division, William H. Bristow, Assistant Superintendent, Bureau of Curriculum Research, and George Grossman, Acting Director, Bureau of Mathematics.

As Chairman of the Junior High School Mathematics Curriculum Committee, Paul Gastwirth, Principal of Edward Bleeker Junior High School, coordinated the work of various committees engaged in planning for the new seventh year mathematics program.

Frank J. Wohlfort, Acting Assistant Director, Bureau of Mathematics, worked with the coordinators and arranged for the experimental tryout of the program in the junior high schools.

Miriam S. Newman, Staff Coordinator, Bureau of Mathematics, served as project leader for the writing of the experimental materials and was the principal writer of the revised materials during the school year 1965-1966 and the summer of 1966.

The members of the Junior High School Curriculum Committee who participated in planning the scope and sequence and in the initial preparation of lesson plans were: Spencer J. Abbott, Florence Apperman, Samuel Bier, Charles Bechtold, Anna Chuckrow, Samuel Dreskin, Charles J. Goode, Helen Halliday, Helen Kaufman, Rose Klein, Miriam S. Newman, Alfred Okin, George Paley, Meyer Rosenspan, Benedict Rubino, Joseph Segal, Ada Sheridan, Murray Soffer, Bertha Weiss, and Frank J. Wohlfort.

Junior High School Mathematics Coordinators and teachers who prepared lesson plans for classroom tryout were: Florence Apperman, Sidney Gellman, Charles J. Goode, Meyer Klein, Morris Leist, Miriam S. Newman, Benedict Rubino, Marilyn Sacco, Murray Soffer, Bertha Weiss, and Frank J. Wohlfort.

Leonard Simon, Junior High School Curriculum Coordinator, Bureau of Curriculum Development, was a member of the original planning group and continued throughout to assist in the planning, coordinating, revising and preparing of the materials for publication.

Teachers and supervisors who used the material in classroom tryout and who played a part in the evaluation and revision include:

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Simon Shulman designed the cover.

Maurice Basseches, Editor, had over-all responsibility for design and production.

# CHAPTER I

The concept of set, introduced in this chapter, will play an important part throughout the course. This section contains materials and suggested procedures for developing the following set concepts and understandings:

- meaning of set
- set language and symbols
- one-to-one correspondence
- sets and numbers
- set operation of union

The term set is used by mathematicians to refer to any well-defined collection of discrete objects. It is actually an undefined term. The elements of a set may be physical objects or abstract ideas.

The concept of set is a simple one, yet it is a powerful mathematical tool. Throughout the course, the idea of set is used to clarify many concepts, including operations and properties of numbers, factoring, the solution of equations and inequalities, and graphs. Geometric as well as arithmetic ideas can be clearly and concisely formulated in terms of sets. Set concepts and terminology learned by pupils in connection with numbers can be applied easily to experiences with sets of points. Thus, ideas of set can serve as a bridge between number ideas and basic geometric ideas.

The fundamental concepts of set and the symbolism and terminology needed to discuss them are learned easily and can arouse considerable pupil interest. Therefore, the simple ideas of set theory provide an excellent motivating topic at the beginning of this course.

The material in this chapter demonstrates that a number is an idea independent of the numerals used to represent it. Procedures are suggested for extending pupils' understanding of ancient systems of numeration; the polynomial form of writing numerals in base ten; a base five numeration system.

Throughout our work with number, a distinction is made between number and numeral. The invention of numerals, symbols for numbers, provides us with a concrete representation for the abstraction called number. When we compute in arithmetic, we are replacing certain numerals by others.

A numeration system is a systematic scheme of denoting or naming numbers. It consists of a set of symbols and some rules for combining the symbols to name various numbers. In their study of numeration systems used by ancient civilizations, pupils are led to see that the numbers used by these civilizations are the same as our own. Only the names are different. Pupils are guided to an appreciation of the advantages of our present system of numeration over ancient ones.

Writing numerals in expanded form using exponents, as, for example, writing

$$2,342 \text{ as} (2 \times 10^3) + (3 \times 10^2) + (4 \times 10) + (2 \times 1)$$

increases the pupil's understanding of base ten notation. This understanding is further enhanced by a study of numeration systems with bases other than ten. Many pupils really understand the grouping techniques in base ten only after they contrast them with corresponding techniques in other bases.

# CHAPTER I

## NUMBERS and NUMERALS

Lessons 1-12

### Lesson 1

Topic: Overview of Sets

Aim: To review and extend some basic concepts about sets

Specific Objectives:

Concept of set
Members or elements of a set
Describing a set ; set notation

Motivation: What do we mean when we say: a set of dishes?

I. Procedure

A. Concept of set

1. Have pupils give examples of the use of the term: set

    a set of trains
    the set of pupils in the class
    the set of numerals on the chalkboard
    a set of airplanes
    a set of numbers
    a set of golf clubs

2. Elicit synonyms for set:

    collection of stamps
    group of children
    family of birds
    flock of sheep

3. Have pupils name some sets whose members are not of the same kind:

    the set of articles on the desk
    the set of objects in a museum

4. Conclude that a set is a collection of objects or ideas.

B. Members or elements of a set

1. Each object in a set is called an element of the set or a member of the set.

2. Name the elements of the set of textbooks you brought to school

today; the elements of the set of implements you brought with you for writing; the members of the set of officers of your class; the members of your set of teachers.

C. Description of a set and set notation

1. A set can be designated in different ways:

a. Descriptive method:

the set of boroughs which make up New York City
the vowels of the English alphabet

b. Listing method:

the set whose members are: Manhattan, Bronx, Brooklyn Queens, Richmond

the set of letters: a, e, i, o, u

Note: We describe sets so that we may know whether or not an object is a member of a set.

2. Set notation

a. Braces: Braces are used to enclose the names of the members of a set. For example, to designate the boroughs of New York City whose names begin with B, we write:

{Bronx, Brooklyn}

This is read: the set whose members are Bronx and Brooklyn.

b. Capital letters: To discuss or name a particular set, we generally use a capital letter.

A = {Manhattan, Bronx, Brooklyn, Queens, Richmond}

This is read: A is the set whose members are: Manhattan, Bronx, Brooklyn, Queens, Richmond.

Note: We do not use the word set before the capital letter.

3. Have pupils consider:

a. which method, listing or descriptive, is used in describing the following sets:

The New England States
{Washington, John Adams, Jefferson}

b. Describe each set in a the second way.

II. Practice

A. Write the names of the members of each set in braces and designate each set with a capital letter.

1. The set of people who are the members of your immediate family.

2. The set of members in your class who play a musical instrument.

3. The set of even numbers from 10 through 20.

4. The set of odd numbers from 19 through 31.

B. Use the descriptive method of designating these sets:

1. $X = \{a, e, i, o, u\}$ (The members of set X are the letters of the alphabet which are vowels.)

2. P = {Alaska, California, Oregon, Washington}

3. $C = \{1, 2, 3, 4, 5\}$

4. $L = \{1, 10, 100, 1000\}$

5. V = {tenths, hundredths, thousandths}

C. Describe three sets by the listing method; describe three sets by the descriptive method.

III. Summary

A. What is a set?

B. What is each object in a collection called?

C. What are two ways of designating a set?

D. What new vocabulary did you learn in this lesson?

(set, member or element of a set, braces)

E. What new symbols did you learn?

## Lessons 2 and 3

**Topic:** Sets and Numbers

**Aim:** To reinforce and extend the meaning of number as a property of sets

**Specific Objectives:**

The meaning of one-to-one correspondence
Number is a property of sets
Comparing numbers (order)
The number of a set; finite, infinite, empty sets
Equivalent sets; equal sets
Subsets

**Motivation:** Jim has arranged chairs for a meeting of the School Library Committee. How can he tell without counting whether there are enough seats for everyone?

### I. Procedure

#### A. One-to-one correspondence

1. Refer to challenge. Elicit that by pairing each committee member with a chair, we can tell whether there are enough seats for everyone, too few seats, or too many.

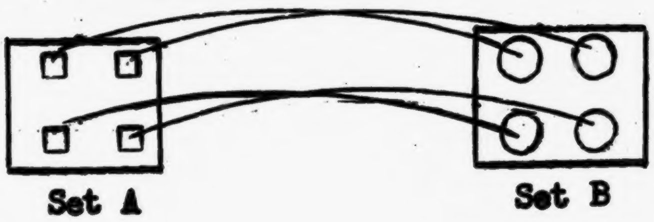
2. If each committee member can be paired with a chair, and each chair can be paired with a committee member, we say there is a one-to-one correspondence between the set of committee members and the set of chairs. In a one-to-one correspondence between two sets, neither set will have any members that have not been paired.

3. Ask pupils to point out other examples of one-to-one correspondence between sets.

   Set of tokens to set of people passing through turnstile
   Set of pencils to set of pupils

#### B. Number is a property of sets

1. Have pupils set up a one-to-one correspondence between set A and set B below.



Note: The diagram shows one of the ways in which each member of Set A is paired with one member of Set B. There are many others.

2. Have them suggest other sets which can be put into one-to-one correspondence with Set A. How many such sets do you think we could imagine? What property have all these sets in common?

3. Elicit that all the sets which can be put into one-to-one correspondence with Set A have the property of "fourness."

4. Tell pupils that when there is a one-to-one correspondence between sets, we say all of these sets have the same (cardinal) number.

5. Have pupils suggest sets which have the same number.

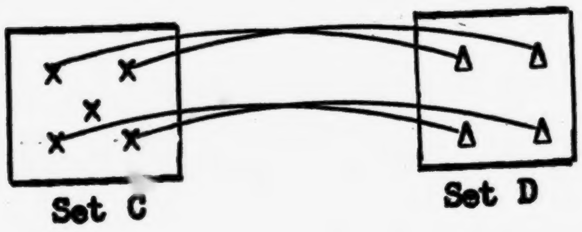
   the set of pupils in the classroom
   the set of mathematics textbooks

6. Have pupils recall that the symbol we use to represent a number is called a numeral.

   What is the numeral for the number of Set A in B-1?

C. Comparing numbers

1. Consider the following two sets:



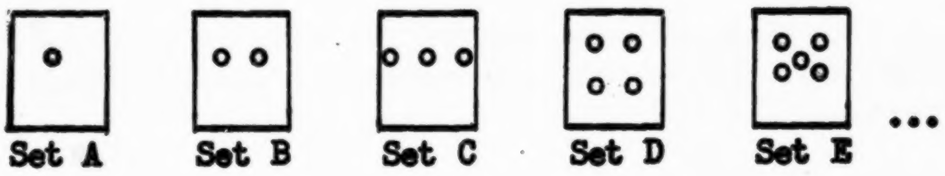
Have pupils try to set up a one-to-one correspondence between these two sets. Elicit that whereas we can pair every member of Set D with a member of Set C, we cannot pair every member of Set C with a member of Set D. There is one member of Set C left over.

2. Tell pupils that when the members of a Set C are paired one-to-one with the members of a Set D, and there is at least one member of C left over, we say the number of C is greater than the number of D. We can also say the number of D is less than the number of C.

3. What is the number of Set C? (5) of Set D? (4)
   Which set has the greater number?

D. The number of a set; finite, infinite, empty sets

1. Have pupils consider the following sets:



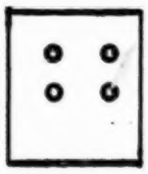
What number is associated with all sets that can be put into one-to-one correspondence with Set A? (1) with Set B? (2) with Set C? (3), etc.

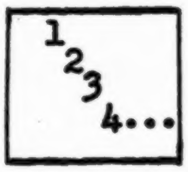
2. Have pupils realize that the numbers associated with the above sets form the set of counting numbers.

   C = {1, 2, 3, 4, 5, ...}

   The three dots indicate that the numbers continue indefinitely in the same pattern. This is read: the set whose members are 1,2,3, and so on.

   The set of counting numbers enables us to tell "how many" there are in any set. The objects of a set and the numerals for the counting numbers, beginning with 1 and proceeding in order, are paired one-to-one until all the members of the set are paired. The last numeral used in the pairing process names the number of members in the set.





3. Have pupils tell the number of pupils present in their class; the number of examples in the set of examples in yesterday's homework assignment; the number of members in the set of counting numbers from 5 through 11.

   a. Have them note that in each case it is possible to count the number of elements in the set with the counting coming to an end. Such a set is therefore called a finite set.

   b. Elicit additional illustrations of finite sets.

4. Can you tell how many members are in C = {1,2,3,4,5,...}?

   a. Have pupils see that the members of Set C cannot be counted with the counting coming to an end. Such a set is called an infinite set.

   b. Elicit additional examples of infinite sets:

      the set of even numbers
      the set of odd numbers

5. How many members are in the set of boys in the class who are over nine feet tall?

   a. Have pupils realize that a set may have no members. Tell pupils that a set which has no members is called the "empty set" or "the null set." The symbol: { } means "empty set." The empty set may also be symbolized by ∅.

   b. Elicit additional illustrations of the empty set,

   c. What is the number of the empty set? (0)

   d. Differentiate between a set which is empty and a set which has zero as its only member.

      1) What is the set of answers to 12-12=? {0}

      2) What is the number of elements in this set? (1)

E. Equivalent vs. equal sets

1. Refer to challenge. Let us assume that each member of the set of committee members has been paired with each member of the set of chairs and there are no unpaired elements in either set. How can we describe such a matching? (one-to-one correspondence)

2. Tell pupils that when the elements of two sets can be put into one-to-one correspondence, the sets are said to be equivalent sets.

3. Elicit that when two or more sets are equivalent, each set contains the same number of elements.

4. Elicit examples of equivalent sets.

5. Consider the following sets:

   A = {a,b,c,d,e}

   B = {c,e,d,b,a}

   Are these sets equivalent? Why?
   What else do you notice about sets A and B? (Both sets contain precisely the same elements although the elements are listed in different order.)

6. Tell pupils that two sets that contain precisely the same elements are said to be equal sets.

7. Elicit examples of equal sets.

## F. Subsets

### 1. Concept of subset

Problem: George has selected a set of three books from the library shelves. The books are members of the set L = {spy story, mystery story, biography}. He is trying to decide which of the three to borrow. What might his decision be?

a. Have pupils list possible sets:

A = {spy story}
B = {mystery}
C = {biography}
D = {spy story, mystery}
E = {spy story, biography}
F = {mystery, biography}
G = {spy story, mystery, biography}

b. If he remembers suddenly that he will be having a number of tests next week, he may decide to take none. Let set H = { }.

c. Have pupils note that every member of set D is a member of set L. The same is true for every set listed in a. We say that a set, such as set D, is a subset of set L, if every member of set D is also a member of set L.

d. Have pupil note set G. Is every member of set G also a member of set L? Conclude that a set may be a subset of itself.

e. Set H is the empty set. Tell pupils that the empty set is considered a subset of every set.

### 2. Recognizing subsets

a. Select the sets which are subsets of the set {0,1,2,3,4,5}.

1) {0,2}
2) {1,2,3,4}
3) {3,4,5,6}
4) {0,1,2,3,4,5}
5) { }
6) {1,3,5,7}
7) {0,2,4,6,8}
8) {0,1,3,4}

b. List three subsets of the set {2,4}

## II. Practice

A. What set of objects in your classroom has the number property of a set of seven? of a set greater than ten?

Explain how you would pair the objects in the set with the set of numerals for the counting numbers to verify your answer.

B. Rearrange these numerals so that the numbers they represent are in order of size. Place the smallest first.

1. 1,2,4,6,5,7,3

2. 2+1, 8-6, 2+3, 9-5, 5+1, 3+4

3. V, X, VII, IX, VIII, XI, VI

C. Suppose 10 tickets were sold and the first one had the numeral 4 on it. What was the numeral on the last ticket, if they were sold in order?

D. Mary read from page 21 through page 34 of her history book last night. How many pages did she read?

E. Which of the following sets have the same number?

1. G = {e,f,g,h,i,j,k}
2. D = {1,2,3,4,5}
3. {John, Jack, Mary, Sue, Ellen}
4. Set of days of the week

F. Tell the number of each set.

{5,8} {1,2, ..., 9} {0} { }

G. Which pairs of the following sets are equivalent?

1. A = {a,b,c,d}
2. B = {0,1,2,3,4}
3. C = {Sue, Jane, Mary, Ann}
4. D = {a,b,c,d,e}

H. Which pairs of the following sets are equal?

1. L = {John, Joe, Jack}
2. M = {△, □, ▭, ○}
3. N = {▭, ○, △, □}
4. {Jack, Joe, John}

I. Tell which of the following sets are finite, which infinite, which empty.

1. The set of odd numbers from 1 through 5.

2. The set of counting numbers greater than 2 and less than 3.

3. The set of all the counting numbers.

J. List all the subsets of {1,2}. How many subsets are there?

## III. Summary

A. What is meant by a one-to-one correspondence between two sets?

B. When do two sets have the same number?

C. When is the number of one set greater than the number of another set?

D. How do we determine "how many" objects there are in a set?

E. What is the difference between a finite set and an infinite set?

F. What is the empty set?
What symbols do we use to show the empty set?

G. Are all equal sets equivalent?
Are all equivalent sets equal?
Explain.

H. What two sets are subsets of every set?

I. What new vocabulary did you learn in this lesson? (finite set, infinite set, empty set, subset)

J. What new symbols did you learn? { }, ∅

## Lesson 4

**Topic:** Systems of Numeration

**Aim:** To extend understanding of the Roman system of numeration

**Specific Objectives:**

To review the basic symbols of Roman numeration

To review the principles used in combining the basic symbols to write numerals for computation

To understand the difficulties of computation with Roman numerals

**Motivation:** Why is the Roman system of numeration used so little today?

I. Procedure

A. Review of basic symbols

1. Review the set of seven basic symbols of the Roman system of numeration:

   I, V, X, L, C, D, M

2. Review the value of each symbol:

| I | V | X | L | C | D | M |
|---|---|---|---|---|---|---|
| 1 | 5 | 10 | 50 | 100 | 500 | 1000 |

3. The value of V is how many times the value of I?
   The value of X is how many times the value of V, etc.
   Have pupils observe the "5 times, 2 times" pattern.

B. Review of principles used in combining basic Roman numerals.

1. What is the value of:

   III     XXX     CC     MM     XXIII

   a. Have pupils note the repetition of symbols.
   b. How do we find the value of a Roman numeral when the basic symbols are repeated? (We find the sum of the values of the individual symbols.)

2. In our system, upon what does the value of each digit in the numeral 444 depend? (its place or position)

   In XXX does the value of each basic symbol depend upon its place or position in the numeral? Explain.

3. What is the value of each digit in 258?
   What is the sum of the values of the digits in 258?
   What is the value of each basic symbol in XVII?

What is the sum of the values of these basic symbols?

4. Elicit that the Roman system is additive as well as repetitive, but does not use place value as we do.

5. What is the value of X?
   What is the value of I?
   What is the value of XI?
   What is the value of IX?

6. Elicit that the Roman system has a subtractive principle. A symbol for a lesser number placed before a symbol for a greater number indicates subtraction of the numbers.

   In this system, it is customary to use:

   only I to the left of V and X
   only X to the left of L and C
   only C to the left of D and M

   The subtractive principle thus applies only to 4 and 9, 40 and 90, 400 and 900, etc.

7. What is the value of these Roman numerals?

| | |
|---|---|
| a. IV (V - 1 = 4) | e. XIV (10 + (5-1) |
| b. IX (X - 1 = 9) | f. XIX (10 + (10-1)) |
| c. XL (L - X = 40) | g. CXL (100 + (50-10)) |
| d. CD (D - C = 400) | h. MCM (1000 + (1000-100)) |

C. Difficulties of computation with Roman numerals

1. Have pupils try to compute the following using Roman numerals. Then have them compute using our system.

   a. Find the sum of MMXLV and MXCVI.
   b. From MCMLII subtract MCXLVI.

2. Elicit that computation with Roman numerals is difficult. Tell pupils that the Romans did not compute as we do. Computation was done on counters. They used their numerals only for recording the results of the computation.

3. Refer to challenge. Have pupils realize that Roman numeration fell into disuse because of the development of a numeration system - the Hindu-Arabic system - that greatly facilitated computation.

## II. Practice

A. Write the Hindu-Arabic numeral for each of the following:

1. CXVII
2. MC
3. DC
4. XLVI
5. MMMCCXLIV
6. DCLXIV
7. CXL
8. CDXIX

B. Write Roman numerals for: 24, 89, 245, 655, 999, 2339.

C. Write in Roman numerals: 30, 80, 110, 505, 800, 1010.

D. Is there a symbol for zero among Roman numerals?

E. Placing a bar above a basic symbol in Roman numerals indicates multiplication by 1000.

What is the value of: $\overline{X}$ $\overline{C}$ $\overline{L}$ $\overline{V}$ $\overline{X}X$?

F. What numeral in the following set does not name the same number as all the others?

{XXXX, XL, XXXV}

G. Which numeral in the following set names an odd number?

{IV, VI, XIII, XX}

## III. Summary

A. What is the set of basic symbols used in the Roman system of numeration?

B. What are some characteristics of Roman numeration? (repetitive, additive, subtractive)

C. What restrictions on subtracting does the Roman system have? (I only before V and X, etc.)

D. Why did the Romans use a counter for their computations?

E. What important symbol is lacking in this system?

## Lesson 5

Topic: The Decimal System of Numeration

Aim: To extend understanding of the Hindu-Arabic System of Numeration

Specific Objectives:

To review the set of basic symbols in Hindu-Arabic numeration: {0,1,2,...9}
To review grouping by tens, ten tens, and so on
To review the idea of place value
To learn to express a number by an expanded numeral

Challenge: Suppose there were no zero in our numeration system. How would you represent the number three hundred nine?

I. Procedure

A. Review of Hindu-Arabic symbols of numeration

1. The set of numerals used in the Hindu-Arabic numeration system is {0,1,2,...9}.

   Elicit that with these ten symbols or digits, we are able to write numerals for the whole numbers less than ten, ten, and numbers greater than ten, that is, for all whole numbers.

2. Review that a symbol for the number zero does not exist in the Roman system of numeration.

B. Our system is based on groups of ten

1. Have pupils count the number of pupils present today in the mathematics classroom, e.g., 34.

   How many sets of ten pupils are there?
   How many pupils are not in a set of ten pupils?
   How does the numeral 34 tell you how the pupils are grouped?

2. In the lunchroom today, there were more than ten sets of ten pupils.
   What is another name for ten sets of ten? (one hundred)

3. Elicit that our system of writing numerals is based on groups of ten. This is why it is called a base ten or decimal numeration system.

C. Place value

1. Have pupils compare the representation of a number such as 325 in Roman numerals and in our numeration system.

   325 CCCXXV

Have them realize that in CCCXXV the value of each basic symbol does not depend upon its place or position in the numeral.

2. Review the names and values of the places in the decimal system of notation.

... Thousands Hundreds Tens Ones

Have pupils recall that as we move to the left in the place value table, the value of each place is ten times that of the place to its right. Thus,

10 = 10x1
100 = 10x10
10000 = 10x1000, etc.

3. Elicit that in our system of numeration the value of a digit in a numeral is the product of the value of the number represented by the value of the place. For example, in the number 325,

the value of the digit 3 is 3x100 or 300
the value of the digit 2 is 2x10 or 20
the value of the digit 5 is 5x1 or 5

What is the sum of the values of the digits in 325?

4. Refer to challenge.
What does each symbol in 309 represent?
Have pupils see that without a symbol for zero, we might find it necessary to represent the number three hundred nine as:

3 - 9, or 3 9

Why is this representation confusing?

D. Expressing a number by an expanded numeral

1. Have pupils represent several numbers as a sum of products in terms of place value.

a. 88 = 8 tens + 8 ones
= (8x10) + (8x1)

b. 257 = 2 hundreds + 5 tens + 7 ones
= (2x100) + (5x10) + (7x1)

c. 4365 = 4 thousands + 3 hundreds + 6 tens + 5 ones
= (4x1000) + (3x100) + (6x10) + (5x1)

2. Tell pupils that when a number is expressed as a sum of products, as in #1, it has been expressed by an expanded numeral. We often write an expanded numeral for standard numerals such as 88, 257, 4365.

3. Have pupils write expanded numerals for:

   a. 23 b. 381 c. 507 d. 6986

4. Have them write standard numerals for the following:

   a. 6 tens + 3 ones

   b. 9 hundreds + 4 tens + 1 one

   c. (3x10) + (2x1)

   d. (7x100) + (4x10) + (2x1)

   e. (6x1000) + (0x100) + (2x10) + (0x1)

II. Practice

A. Using the digits 1, 3, 4 once and only once, write numerals for as many numbers as you can. (134, 314, 431, etc.)

B. Select three of the numbers for which you have written numerals in the preceding example. For each, give the value of each digit in the numeral.

C. Consider the numeral 44,444.

   1. The value of the digit 4 in tens place is how many times the value of the digit 4 in ones place?

   2. The value of the digit 4 in hundreds place is how many times the value of the digit 4 in tens place?

   3. The value of the digit 4 in thousands place is how many times the value of the digit 4 in tens place?

D. Replace each frame so that a true statement results.

   1. 482 = (□ x 100) + (△ x 10) + (○ x 1)

   2. 6045 = (▭ x 1000) + (□ x 100) + (△ x 10) + (○ x 1)

E. Write an expanded numeral for each of the following:

   1. 312 2. 1907 3. 25,430 4. 678,916

F. Write the standard numeral for each of the following:

1. (4 x 1000) + (5 x 100) + (2 x 10) + (8 x 1)
2. (9 x 10,000) + (0 x 1000) + (3 x 100) + (1 x 1)
3. (7 x 100,000) + (5 x 10,000) + (9 x 1000) + (0 x 100) + (4 x 1)

G. The diameter of the planet Uranus is approximately 30,900 miles. Express this number as an expanded numeral.

III. Summary

A. In our base ten numeration system, how many digits are used to name the base number ten?

B. In our base ten system, how many digits are needed to name any whole number?

C. What does the value of a digit in a numeral depend upon?

D. What is meant by an expanded numeral for a number?

E. Why is our numeration system called a decimal system?

F. What new vocabulary did you learn today?

(standard numeral, expanded numeral)

## Lessons 6 and 7

Topic: The Decimal System of Numeration

Aim: To express place value using exponents

Specific Objectives:

To learn the meaning of powers of ten
To learn the meaning of exponent
To express a number by an expanded numeral using exponents

Challenge: How can you express the number 1,000,000 in a shorter way?

I. Procedure

A. Meaning of powers of ten

1. Elicit that 100 can be expressed as 10x10.
   1000 can be expressed as 10x10x10.
   10,000 can be expressed as 10x10x10x10, etc.

2. Tell pupils that numbers such as 100, 1000, 10,000 are called powers of ten. They are the product of tens only.

3. Have pupils note that 10 is used as a factor twice in 10x10; 10 is used as a factor three times in 10x10x10, and so on.

B. Meaning of exponent

1. Instead of writing 100 as 10x10, we can write $100 = 10 \times 10 = 10^2$
   Instead of writing 1000 as 10x10x10, we can write $1000 = 10 \times 10 \times 10 = 10^3$

2. Tell pupils that the symbols "$^2$" and "$^3$" are called exponent symbols (exponents, for short). They tell us the number of times 10 is used as a factor.

3. Replace the frames.

   a. $10{,}000 = 10 \times 10 \times 10 \times 10 = 10^{\square}$

   b. $100{,}000 = 10 \times 10 \times 10 \times 10 \times 10 = \square^{5}$

   c. $1{,}000{,}000 = \square \times \square \times \square \times \square \times \square \times \square = \square^{\triangle}$ (challenge)

4. Since $100 = 10^2$, we say that 100 is the second power of ten.

   a. What is the third power of ten? the fourth power?

   b. Which power of ten is 100,000? 1,000,000?

5. Have pupils realize that we can use exponents to show the number of times numbers are used as factors.

a. $2^3 = 2x2x2$, or 8; 8 is the third power of 2.

b. $7^2 = 7x7$, or 49; 49 is the second power of 7.

c. $3^4 = 3x3x3x3$, or 81; 81 is the fourth power of 3.

d. 36 is the second power of _____?

C. Expressing a number by an expanded numeral using exponents

1. Complete the table:

| | | | |
|---|---|---|---|
| 400 | 4x100 | 4x10x10 | $4x10^2$ |
| 6,000 | 6x1000 | 6x10x10x10 | ? |
| 70,000 | 7x10000 | ? | ? |
| 800,000 | ? | ? | ? |

2. Write in expanded form using exponents:

8,341 = (8x1000) + (3x100) + (4x10) + (1x1) (think)
= $(8x10^3)$ + $(3x10^2)$ + (4x10) + (1x1) (write)
605 = $(6x10^2)$ + (0x10) + (5x1)
79,214 = $(7x10^4)$ + $(9x10^3)$ + $(2x10)^2$ + (1x10) + (4x1)

II. Practice

A. Replace the frames.

1. $10x10 = 10^{\square}$
2. $10x10x10x10 = 10^{\square}$
3. $100 = \square^3$
4. $10,000 = \square^4$
5. $100,000 = \square^{\Delta}$
6. $10,000,000 = \square^{\Delta}$
7. $5x5 = 5^{\square}$
8. $4x4x4 = \square^3$
9. $16 = 4^{\square}$
10. $16 = 2^{\square}$

B. Write the standard numeral for each of the following:

1. $10^3$ 3. $10^4$ 5. $6^2$ 7. $10^8$ 9. $10 \times 10^2$

2. $10^5$ 4. $2^3$ 6. $10^6$ 8. $5^3$ 10. $2 \times 2^2$

C. What is the value of $10^4$? of $6 \times 10^2$? of $23 \times 10^3$?

D. Rewrite using exponents.

1. 500 ($5 \times 10^2$) 3. 90,000

2. 8000 4. 15,000,000

E. Write in expanded form using exponents.

1. 255 3. 4,027

2. 3698 4. 18,529

F. What is the second power of 10? the fifth power of 10?

G. What is the third power of 3? the fourth power of 3?

H. Which power of 10 is 10,000? Which power of 7 is 49?

I. Find the second power of each of the following numbers:

1. 2 2. 5 3. 7 d. 11

J. Which is greater, $4^3$ or $3^4$? Explain.

III. Summary

A. What is meant by powers of ten? (numbers which are the product of tens only)

B. What is meant by powers of three? powers of five?

C. Give two examples of powers of ten. (100, 10,000, etc.)

D. What is the meaning of an exponent?

E. Write a four-digit numeral for a number.
Express the number by an expanded numeral; by an expanded numeral using exponents.

F. What new vocabulary have you learned today?

(power, exponent)

## Lesson 8

Topic: The Decimal System of Numeration

Aim: To extend the understanding of place value through hundred billions

Specific Objectives:

To extend the place value table through hundred billions
Reading and writing word names for numbers

Challenge: How many times a million is a billion?

I. Procedure

A. Place value through hundred billions

1. Review place value through millions. Set up place value table.

| Millions | Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Ones |
|---|---|---|---|---|---|---|
| 1,000,000 | 100,000 | 10,000 | 1000 | 100 | 10 | 1 |

2. Recall plan for assigning place value to a digit in a numeral and extend place value table through **hundred billions.**

| Hundred Billions | Ten Billions | Billions | Hundred Millions | Ten Millions | Millions | Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Ones |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 100,000,000,000 | 10,000,000,000 | 1,000,000,000 | 100,000,000 | 10,000,000 | 1,000,000 | 100,000 | 10,000 | 1000 | 100 | 10 | 1 |

a. How many times a thousand is a million?
b. How many times a million is a billion? (Challenge)

Note: In the United States, a billion is a thousand millions; in Great Britain, a billion is a million millions.

3. Elicit that the place value names for tens place, hundreds place, thousands place, etc. may be expressed as: 10, $10^2$, $10^3$, $10^4$, and so on, through $10^{11}$ for one hundred billion.

B. Reading and writing word names for numbers

1. Have pupils recall that when reading numerals, we read in groups, called periods. We use commas to separate the periods.

2. Elicit name of each period: units, thousands, millions, billions. Elicit that each period contains ones, tens and hundreds.

| Billions | | | Millions | | | Thousands | | | Ones | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Hundred Billions | Ten Billions | Billions | Hundred Millions | Ten Millions | Millions | Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Ones |
| $10^{11}$ | $10^{10}$ | $10^9$ | $10^8$ | $10^7$ | $10^6$ | $10^5$ | $10^4$ | $10^3$ | $10^2$ | 10 | |
| | 1 | 3 | 4 | 9 | 6 | 0 | 2 | 1 | 6 | 8 | 5 |

3. Have pupils recall that to read a numeral we read the part of the numeral (starting at the left) in each period, adding the period name. We do not read the word "ones" at the end. For example, we read the numeral 13,496,021,685 in this way:

   thirteen billion, four hundred ninety-six million, twenty-one thousand, six hundred eighty-five.

2. Read the numeral: 263,409,696,378 the same way.

3. Write the standard numeral that represents two hundred thirty billion, 6 hundred seven million, three hundred ten thousand, forty-six. (If necessary, have pupils use the period chart to do this.)

II. Practice

A. Count as directed below:

1. by 100,000's to 1,000,000
2. by 10,000,000's to 100,000,000
3. by 100,000,000's to 1,000,000,000

B. Read the following numerals:

1. 87,493
2. 325,671
3. 202,305
4. 4,000,000
5. 1,849,217
6. 50,300,000
7. 134,560,000
8. 14,000,000,000
9. 17,500,000,000
10. 22,356,000,000

C. Write the standard numerals for these numbers.

1. Eight thousand, four hundred fifty
2. Two hundred fifty thousand, nine hundred seventeen
3. Three hundred seven million, one hundred eighty-five thousand
4. Five billion, six hundred ninety-seven million
5. One hundred four billion

D. Express each of the following using exponents:

1. 36,000 ($36 \times 10^3$)
2. 67,000,000
3. 142,000,000
4. 92,000,000,000
5. 10,000,000,000

E. How many thousands are there in $\frac{1}{2}$ million? in $\frac{1}{4}$ million?

F. How many millions are there in $\frac{1}{2}$ billion? in $\frac{1}{4}$ billion?

G. Change numerals to word names and word names to standard numerals.

1. The planet Venus is about 67,270,000 miles from the sun.
2. In 1964, the gross national product was approximately six hundred eighteen billion, five hundred million dollars.
3. A satellite sent a message from 34,000,000 miles in space.
4. On the stock market, eight million, four hundred thirty-two thousand, nine hundred sixty-two shares were traded.
5. Congress appropriated $2,485,000,000 for education.

H. How many times as large as the first number is the second number?

1. 20; 200
2. 35; 3500
3. 143; 14,300
4. 205; 2,050,000
5. 1190; 119,000,000
6. 15,000; 1,500,000,000

III. Summary

A. How many times ten million is a billion?

B. What are the names of the first four periods?

C. How do we read a numeral?

D. What is the least number that can be named in any period?
What is the greatest?

## Lesson 9

Topic: Other Numeration Systems

Aim: To express a number in various bases

Specific Objectives:

To extend understanding of the meaning of numeral
To arrange a set of objects in various groups; to express these groupings by numerals in the appropriate base

Motivation: It is believed that our decimal system of numeration is based on the number ten because we have ten fingers. How would we express our numbers if our system were based on the number eight?

I. Procedure

A. Extend the understanding of the meaning of numeral

1. Review that a numeral is a symbol for a number.
The number 25 may be expressed as:

25 XXV $5^2$ $(2\times10) + (5\times1)$

Elicit that these numerals all name the same number.

2. Have pupils express several other numbers using a variety of numerals for each one.

B. Groupings of objects; the use of numerals to name the groupings

1. Have pupils consider a set of x's such as the set shown below:

x x x x x x x x x x x x x x x x x

Have them group the elements of the set by tens.

$\boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}\ x\ x\ x\ x\ x\ x\ x$

a. How many groups of ten elements are there? (1)
How many elements are there not in a group of ten? (7)

| tens | ones |
|---|---|
| 1 | 7 |

b. Elicit that the numeral 17 in our base ten system shows this grouping of 1 ten and 7 ones.

2. Have pupils group the elements of the set by eights.

$\boxed{x\ x\ x\ x\ x\ x\ x\ x}\ \boxed{x\ x\ x\ x\ x\ x\ x\ x}\ \ x$

**A system of grouping by eights is called a base eight system.**

a. How many groups of eight are there? (2)
How many elements are there not in a group of eight? (1)

| eights | ones |
|---|---|
| 2 | 1 |

b. In base ten, 1 group of ten and 3 ones can be called 1 ten + 3 ones.
What can we call 2 groups of eight and 1 one in base eight?

2 eights + 1 one

c. A short way of expressing 2 eights and 1 one is 21, base eight. This is read: two, one, base eight, not, twenty-one, base eight.

1) In 17, base ten, what does the digit 1 tell you?
2) In 21, base eight, what does the digit 2 tell you?
3) In 17, base ten, what does the digit 7 tell you?
4) In 21, base eight, what does the digit 1 tell you?

d. Tell pupils that when we want to show in what base a numeral is written, we usually write $21_{eight}$, instead of 21, base eight.

3. Have pupils regroup the seventeen elements in various ways.

a. Groups of seven: $\boxed{x\ x\ x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x\ x\ x}$ x x x

| sevens | ones |
|---|---|
| 2 | 3 |

The base seven numeral which expresses the number of elements is $23_{seven}$ (read: two, three, base seven)

b. Groups of six: $\boxed{x\ x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x\ x}$ x x x x x

| sixes | ones |
|---|---|
| 2 | 5 |

The base six numeral which expresses the number of elements is $25_{six}$.

c. Groups of five: $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ x x

| fives | ones |
|---|---|
| 3 | 2 |

The base five numeral which expresses the number of elements is $32_{five}$.

d. Groups of twelve: $\boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x\ x\ x}$ x x x x x

| twelves | ones |
|---|---|
| 1 | 5 |

The base twelve numeral which expresses the number of elements is $15_{\text{twelve}}$.

4. Elicit that: $\{17, 21_{\text{eight}}, 23_{\text{seven}}, 25_{\text{six}}, 32_{\text{five}}, 15_{\text{twelve}}\}$ consists of different numerals for the same number.

5. Tell pupil that any numeral written without a base indicated is understood to be written in base ten, the base of our decimal system of numeration.

   Note: At this time, do not group any number of objects which requires a numeral of three places. Extension to third and fourth places will be considered in the next lesson.

6. Have pupils determine what number would be represented by the symbol 42 if 42 is interpreted in the following bases:

   $42_{\text{eight}}$ (4 eights + 2 ones, or 34)

   $42_{\text{seven}}$ (4 sevens + 2 ones, or 30)

   $42_{\text{six}}$ (4 sixes + 2 ones, or 26), and so on

II. Practice

A. Regroup twenty objects and record the numeral which shows a grouping by:

1. tens
2. nines
3. eights
4. sevens
5. sixes
6. fives

B. Read the numerals you recorded in A.

C. Write a base ten numeral to represent each of the following:

1. $24_{\text{eight}}$ (2 eights + 4 ones or 20)
2. $87_{\text{nine}}$
3. $54_{\text{six}}$
4. $23_{\text{twelve}}$
5. $44_{\text{five}}$

D. Which statements are true and which are false?

1. $3_{six} = 3_{nine}$
2. $30_{six} = 30_{nine}$
3. $14_{five} < 12_{six}$
4. $37_{nine} > 44_{five}$
5. $22_{seven} = 16_{ten}$

## III. Summary

A. In our base ten system of numeration, we group by tens.
How do we group in a base eight system? in a base twelve system?

B. What are several ways involving different bases in which the number sixteen can be expressed by a numeral?
Does the number change? Explain.

C. If you want to express thirteen in base five, how many groups of five will you have?
How many ones will you have?
Express thirteen with a base five numeral.

D. If a numeral has no indicated base, what base is understood?

## Lessons 10 and 11

**Topic: Numeration in Base Five**

**Aim: To learn the base five (quinary) system of numeration**

**Specific Objectives:**

To compare number of basic symbols in base ten and in base five
To develop a place value table for base five
To interpret numerals in base five by means of place value
To compare base ten and base five numeration systems

**Challenge: When does "100" not mean one hundred?**

**I. Procedure**

A. Comparison of symbols in base ten and in base five

1. Review the set of basic symbols used to express numbers in base ten: {0, 1, 2, ..., 9}

2. Elicit that a base five system would have five symbols: 0, 1, 2, 3, 4.

   Note that *there is no numeral 5 in base five.*

B. A place value table for base five

1. Review place value table for base ten.

Base Ten Place Value Table

| . . . | Hundreds or Ten Tens | Tens | Ones |
|---|---|---|---|
| | 10 x 10 | 10 | 1 |

   a. Write 368 as an expanded numeral.
   b. Elicit that in a base ten numeration system the value of each place is ten times the value of the place to its right.

2. Have pupils consider the following group of x's:

   x x x x x x x x x x x x x x x x x x

   a. How many groups of five x's can you form from the x's?
   b. Have pupils show grouping by fives.

   $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ x x x

c. Have them write the base five numeral which expresses the number of x's.

3 fives + 3 $\quad 33_{\text{five}}$

1) In $33_{\text{five}}$, what does the 3 at the right tell us?
2) In $33_{\text{five}}$, what does the 3 at the left tell us? What is its value?

3. Begin development of a place value table for base five.

a. Base Five Place Value Table

| Fives | ones |
|---|---|
| 3 | 3 |
| 3 fives | 3 ones |

$33_{\text{five}}$

b. Elicit that the base five numeration system is a place-value system. The place at the right is ones place. The next place to the left is fives place.

c. Have pupils use a place-value table and replace each frame so that a true statement results.

1) $12_{\text{five}}$ = ☐ fives + 2 ones
2) $34_{\text{five}}$ = 3 fives + ☐ ones
3) $44_{\text{five}}$ = ☐ fives + ☐ ones
4) $3_{\text{five}}$ = ☐ fives + 3 ones

d. Have pupils compare the place value tables for base ten and base five shown below:

Base Ten Place Value Table

| Tens | Ones |
|---|---|

Base Five Place Value Table

| Fives | Ones |
|---|---|

How are these tables alike? How are they different?

e. Have pupils write the numerals in base five notation for the numbers from one to twenty-four.

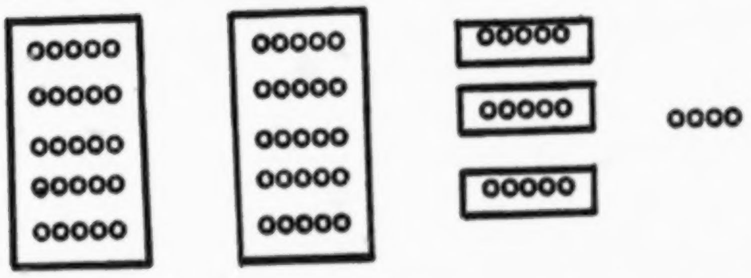
1, 2, 3, 4, 10, 11, 12, ..., 44

4. Extending the place value table for base five

a. Elicit that in base ten we form large groups of ten sets of ten objects.

What is the base ten numeral for the number in ten sets of ten objects? (100)

b. Have pupils examine the grouping of circles below:



1) Have pupils see that whereas in base ten we form a large group from ten sets of ten objects, in base five we form a large group from five sets of five objects.

2) How many large groups of five fives are there?
   How many groups of five circles (not in a group of five fives are there?
   How many circles are not in a group of five?

3) Elicit that the total number of circles is:
   2 five fives + 3 fives + 4 ones or
   2 twenty-fives + 3 fives + 4 ones.

4) In base five, how can we write 2 twenty-fives + 3 fives + 4 ones in short form? ($234_{five}$)

c. Extend place value table for base five to twenty-fives place.

1) Recall that in a base ten numeration system, the place to the left of tens place is ten tens place, or hundreds place.

2) Elicit that in a base five numeration system, the place to the left of fives place is five fives place or twenty-fives place.

3) Have pupils examine the place value tables for base ten and base five as follows:

**Base Ten Place Value Table**

| Ten Tens<br>Hundreds | Tens | Ones |
|---|---|---|

**Base Five Place Value Table**

| Five Fives<br>Twenty-fives | Fives | Ones |
|---|---|---|

How are these tables alike? How are they different?

4) What is the value of $100_{five}$? (Refer to challenge.)

d. Use similar procedures to develop one hundred twenty-fives place; six hundred twenty-fives place.

C. Interpreting numerals in base five by means of place value

Have pupils use place value to change a base five numeral to an equivalent base ten numeral.

1. $432_{five}$ = 4 twenty-fives + 3 fives + 2 ones
= 100 + 15 + 2
= 117

2. $1241_{five}$ = 1 one hundred twenty-five + 2 twenty-fives + 4 fives + 1 one
= 125 + 50 + 20 + 1
= 196

3. $32241_{five}$ = 3 six hundred twenty-fives + 2 one hundred twenty-fives + 2 twenty-fives + 4 fives + 1 one
= 1875 + 250 + 50 + 20 + 1
= 2196

D. Have pupils compare base ten and base five numeration systems.

1. Recall that in the base ten numeration system, grouping is done by tens, ten tens, and so on.
Elicit that in the base five numeration system, grouping is done by fives, five fives, and so on.

2. In the base ten place value system, the value of each place is ten times the value of the place to its right.
In the base five place value system, the value of each place is five times the value of the place to its right.

II. Practice

A. Draw dots and encircle groups to show the meaning of each numeral.

$12_{five}$ $23_{five}$ $44_{five}$

B. Count in base five from five to twelve; from twenty-two to twenty-six; from seventy-three to seventy-seven.

C. Write the base five numerals for the numbers from $434_{five}$ through $1004_{five}$ .

D. What is the value of the numeral 10 in base ten?; in base five? Why do we refer to "10" as the base number?

E. Change each of these numerals to an equivalent base ten numeral.

1. $24_{five}$
2. $421_{five}$
3. $3330_{five}$
4. $4401_{five}$
5. $30003_{five}$
6. $44444_{five}$

F. Which of the members of this set represent odd numbers?

$\{14_{five}, 32_{five}, 23_{five}, 401_{five}, 42_{five}\}$

G. In the following sequence, each number after 1 is three greater than the previous number. Write the names for the next five numbers.

$1, 4, 12_{five}, 20_{five}, \ldots$

III. Summary

A. What are the basic symbols in the base ten system of numerals? in base five?

B. Why is $374_{five}$ meaningless?

C. How many numerals can you write with the basic symbols in base ten? in base five?

D. What is the numeral for the base number in base ten? in base five? in any base?

E. What are the names of the place values in the base five place value table beginning at the ones place and moving to the left?

F. How does the value of each place compare with the value of the place on its right in the base five system?

G. How do we find a base ten numeral for a base five numeral such as $333_{five}$?

## Lesson 12

Topic: Base Five

Aim: To extend understanding of base five

Specific Objectives:

To express base five numerals in expanded form using exponents
To convert from base ten to base five

Challenge: We know that $100_{five}$ represents the number 25.
What numeral in base five represents the number 100?

I. Procedure

A. Expanded form of a base five numeral using exponents

1. Have pupils recall that $25 = 5x5 = 5^2$
$125 = 5x5x5 = 5^3$
$625 = 5x5x5x5 = 5^4$ , and so on.

2. Have pupils express base five numerals in expanded form using exponents.

a. $432_{five}$ = 4 twenty-fives + 3 fives + 2 ones
$= (4x25) + (3x5) + 2$
$= (4x5^2) + (3x5) + 2$

b. $3144_{five}$ = 3 one hundred twenty-fives + 1 twenty-five + 4 fives + 4
$= (3x125) + (1x25) + (4x25) + 4$
$= (3x5^3) + (1x5^2) + (4x25) + 4$

c. $4033_{five}$ = 4 one hundred twenty-fives+0 twenty-fives+3 fives+3
$= (4x125) + (0x25) +(3x5) +3$
$= (4x5^3) + (0x5^2) + (3x5) +3$

d. $24331_{five}$ = 2 six hundred twenty-fives+4 one hundred twenty-fives + 3 twenty-fives + 3 fives + 1
$= (2x625) + (4x125) + (3x25) + (3x5) + 1$
$= (2x5^4) + (4x5^3) + 3x5^2) + (3x5) + 1$

B. Converting from base ten to base five

1. Refer to place value table in base five and elicit that when we express numbers as base five numerals, we must think in terms of one hundred twenty-fives or $5^3$, twenty-fives or $5^2$, fives, and ones.

2. Consider 69.
   How do we record this as a base five numeral?
   Think of a set of 69 objects.

   a. $69 > 25$ and $69 < 125$. We therefore begin with twenty-fives.

   b. How many sets each containing twenty-five elements can be formed from the set of 69 objects? (2)

      We then place 2 in twenty-fives place.

      $69 = 2\_\ \_{}_{five}$. How many elements are left? (19)

   c. How many sets each containing five elements can be formed from a set of 19 objects? (3)

      How many elements are left? (4)

      We then place 3 in fives place and 4 in ones place

      $69 = 234_{five}$

   d. Have pupils check by expanding $234_{five}$.

      $$\begin{aligned} 234_{five} &= (2x25) + (3x5) + 4 \\ &= 50 + 15 + 4 \\ &= 69 \end{aligned}$$

3. Refer to challenge.
   Convert 100 to a base five numeral.

   a. $100 > 25$ and $100 < 125$. Begin with twenty-fives.

   b. There are 4 sets of 25 in 100. Enter 4 in twenty-fives place.

      $100 = 4\ \_\ \_\ {}_{five}$

      How many elements are left? (none)

   c. $100 = 400_{five}$

d. Check by expanding.

$$\begin{aligned} 400_{five} &= (4x25) + (0x5) + 0 \\ &= 100 \quad + \quad 0 \quad + 0 \\ &= 100 \end{aligned}$$

II. Practice

A. Write in expanded form using exponents:

| | |
|---|---|
| 1. $23_{five}$ | 4. $1030_{five}$ |
| 2. $301_{five}$ | 5. $2004_{five}$ |
| 3. $342_{five}$ | 6. $3421_{five}$ |

B. Give the equivalent standard base ten numerals for the numerals in A.

C. Between which two successive powers of five is each of the following:

| | |
|---|---|
| 1. 36 ($36 > 25$ and $36 < 125$) | 4. 500 |
| 2. 87 | 5. 720 |
| 3. 127 | |

D. Convert each of the base ten numerals in C to base five numerals. Check your answers.

III. Summary

A. What are the first four place values in base five numeration beginning at the ones place and moving left?

B. Express these place values of A using exponents.

C. Expand $124_{five}$ using exponents.

D. How would you find the base five numeral for 87?

# CHAPTER II

The major objective of this chapter is to suggest procedures for helping the pupil extend his understanding of the operations and properties of whole numbers, thereby increasing his proficiency in computational skills and in problem solving.

A meaningful approach to the teaching of mathematics demands that considerable emphasis be placed upon the structure of our number system. Thus, the principles involved in the commutative, associative, and distributive laws of mathematics are a fundamental part of this course. Through the understanding and application of these principles, pupils are assisted in developing not only skill but also concepts of the nature of the operations, and understandings underlying the technique of computation. The significance of these laws and properties of mathematics becomes clear to pupils who have opportunities to discover them and to observe what happens when they are used. For example, understanding of a distributive property which combines addition and multiplication gives the pupil increased insight into the algorithm used in multiplication of numbers with two or more places. Thus,

$$\begin{array}{rl} 25 & 3 \times 25 = 3 \times (20 + 5) = (3 \times 20) + (3 \times 5) \\ \underline{\times 3} & \qquad\qquad\qquad\qquad\quad = \quad 60 \quad + \quad 15 \quad = 75 \\ 75 & \end{array}$$

$$\begin{array}{rl} 236 & 4 \times 236 = 4 \times (200 + 30 + 6) \\ \underline{\times 4} & \qquad\qquad = (4 \times 200) + (4 \times 30) + (4 \times 6) \\ 944 & \qquad\qquad = 800 \quad + \quad 120 \quad + \quad 24 \\ & \qquad\qquad = 944 \end{array}$$

An understanding of the meaning of the basic computational processes can enhance the analysis of problem solving. In this connection, the concepts and terminology of sets can be used to clarify the arithmetic operations of addition, subtraction, multiplication and division. When addition is perceived by the pupil as related to the joining of sets, multiplication to the joining of sets of equal size, and so on, his ability to analyze a social situation to determine the appropriate computational process is increased.

# CHAPTER II

## OPERATIONS and PROPERTIES of WHOLE NUMBERS

Lessons 13-28

### Lesson 13

Topic: Addition of Whole Numbers

Aim: To show how the union of sets is related to the addition of whole numbers

Specific Objectives:

To recognize pairs of sets as disjoint sets
To develop the concept of union of two sets
To show how the union of two disjoint sets is related to the sum of two numbers

Challenge: Consider the two sets:

A = {Harry, Mark, Alan, Jim}

B = {Sue, Helen, Flora}

Which members of set A are also members of set B?

I. Procedure

A. Disjoint sets

1. Elicit that sets A and B have no members in common.

2. Tell pupils that when two sets have no element in common they are called disjoint sets.

3. Which of the following pairs of sets are disjoint sets? Explain.

a. R = {1,2,3}  S = {2,5,6,7}

b. M = {a,b,c,d,e}  N = {d,e,f,g,h}

c. P = {0,2,4,6}  Q = {1,3,5,7,9}

d. The set of whole numbers and the set of even numbers.

B. Union of sets

1. Consider the following sets:

   Set E is a class committee for magazines and newspapers.
   Set H is a class committee for books.

   M = {Robert, Laura}

   B = {Nita, Paul, Leonard}

   a. The magazine committee and the book committee were combined to form the class library committee.

      If we call the library committee set L, who are the members of set L?

      L = {Robert, Laura, Nita, Paul, Leonard}

   b. Are there any members of set L not in either set M or set B?

   c. Are there any members of set L who are in both set M and set B?

2. Consider the sets:

   R = {1,2,3,4,5}

   S = {4,5,6,7}

   a. If you combine the members of set R with the members of set S, what is the resulting set?

      T = {1,2,3,4,5,6,7}

   b. Are there many members of set T not in either set R or set S?

   c. Are there any members of set T which are in both set R and set S?

3. Tell pupils that when two sets are "joined" to form a third set, the resulting set is called the union of the two sets.

   The union of two sets is a set which consists of only those elements which belong to either or both sets.

   Set L is the union of set M and set B.
   Set T is the union of set R and set S.

4. Although an element may be in both sets, it appears only once in the union of the two sets.

   Why are there only 7 elements in set T?

5. The symbol for union is $\cup$ .

   $A \cup B$ means the union of set A and set B.

   We read this:"A union B"or"the union of set A and set B."

   $C = A \cup B$ means set C is the union of set A and set B.

6. Name the members of the union of each of these pairs of sets.

   a. D = {John, Joe, Sue}     E = {Ann, Sue, Ellen, Harry}
   b. R = {a,b,c,d,e}     S = {c,d,e,f,g}
   c. V = {0,1,2,3,4}     W = {0,5,10,15}
   d. A = {0,1,2,3,4,...}     B = {1,3,5,7,9,...}

C. Relating the union of disjoint sets to the sum of two numbers.

1. Have pupils consider:

   H = {Frances, Helen, Janet, Beth} and

   K = {Cliff, Gary, Barbara}

   a. Elicit that sets H and K are disjoint sets.
   b. How many elements are in set H? (4)
   c. How many elements are in set K? (3)
   d. Find $H \cup K$.
   e. How many elements are in $H \cup K$? (7)

2. Consider C = {a,e,i,o,u} and D = {1,2,3,4} .

   a. How many elements are in set C? (5)
   b. How many elements are in set D? (4)
   c. Find $C \cup D$.
   d. How many elements are in $C \cup D$? (9)

3. Set M consists of 40 history books.
   Set Q consists of 38 mathematics books.
   How many books will the union of the two sets contain? Explain.

4. Elicit that the sum of the number of elements in a pair of disjoint sets is the number of elements in the union of the two sets.

   The union of two disjoint sets is related to the sum of two numbers.

5. Consider A = {1,2,3,4,5} , B = {4,5,6}

   a. Elicit that these sets are not disjoint.
   b. How many elements are in set A? (5)
   c. How many elements are in set B? (3)
   d. What is A U B? {1,2,3,4,5,6}
   e. How many elements are in A U B? (6)
   f. Why is the number of elements in A U B not the sum of the number of elements in set A and in set B?

II. Practice

A. Which of the following pairs of sets are disjoint sets?

1. {0,1,2,3,5} and {4,6,9,10}

2. {a,f,g,h} and {h,k,m,n}

3. {100,200,300,400} and {500,600,700}

4. The set of whole numbers and the set of odd numbers.

5. The set of even numbers and the set of whole numbers.

B. List the elements of each set.

1. {Mary, Frances, Helen, Rosa} U {Helen, Rosa, Ann}

2. {1,3,5,7} U {1,4,6,8}

3. {3,6,9,12,15} U {1,2,3,4,5}

4. {0,1,2,3,4,...} U {0,2,3,6,8,...}

5. {0,2,4,6,8,...} U {1,3,5,7,...}

6. { } U 1,2,3,4}

C. If A is the set of letters in baseball and B is the set of letters in sport, what is A U B?

D. Construct two sets which are disjoint sets. Then list in braces the members of the union of the two sets.

E. Form two disjoint sets of which the set {2,3,4,5,7} is the union. Form two sets which are not disjoint of which the set {2,3,4,5,7} is the union.

F. Find the number of members in the union of the two sets in each of the following cases:

1. Set A contains 38 notebooks.
   Set B contains 43 pencils.

2. Set R contains 128 science books.
   Set D contains 319 history books.

3. Set P contains 786 pupils.
   Set T contains 28 teachers.

III. Summary

A. What is meant by disjoint sets?

B. What is meant by the union of two sets?
What symbol is used to represent union?

C. How does the number of elements in the union of two disjoint sets compare with the number of elements in each of the given sets?

D. How does the number of elements in the union of two sets which are not disjoint compare with the sum of the number of elements in each of the given sets?

E. What new vocabulary have you learned today?

(union, disjoint set)

Lesson 14

Topic: Addition of Whole Numbers

Aim: To understand the closure and commutative properties of addition

Specific Objectives:

To understand:
- The set of whole numbers is closed under addition
- The set of whole numbers is not closed under subtraction
- The order of adding two whole numbers does not affect the sum: commutative property of addition
- Subtraction is not commutative
- Zero is the identity element of addition

Challenge: When we compute the sum in an addition example, why do we often check our work by adding in the "opposite direction"?

I. Procedure

A. The closure property of addition of whole numbers

1. Review that the set of whole numbers consists of the set of natural numbers and zero.

2. Consider the set: $\{0,1,2,3,4\}$.

   a. Add any two numbers of the set. The addends may be the same number. Is the sum always a member of the set?

      For example, $0+2=2$ (2 is an element of the set)
      $1+3=4$ (4 is an element of the set)
      $1+4=?$

      We do not have 5 in our set so that the sum of two of the numbers of our set is not in our set.

   b. Conclude that the sum of two elements of a set is not always an element of the set.

3. Consider the set: $\{0,1,2,3, \ldots\}$.

   a. What is the sum of 8 and 7? Is the sum a whole number?
   b. What is the sum of 500 and 900? Is the sum a whole number?
   c. After several such illustrations, have pupils conclude that for any two whole numbers there is a sum which is also a whole number.

4. Tell pupils that since the sum of any two elements of the set of whole numbers is an element of the set, we say the set of whole numbers is closed under the operation of addition. This is called the Closure Property of Addition for Whole Numbers.

B. The set of whole numbers is not closed under subtraction.

1. Consider the set: {0,1,2,3, ...}

   a. Can we always subtract two whole numbers? Is the difference always a whole number?

      10-3 = 7
      7-6 = 1
      150-28= 122
      13-16= ?

   b. Have pupils conclude that the difference of two whole numbers is not always a whole number.

2. Elicit that the set of whole numbers is not closed under subtraction.

3. Have pupils see that one counterexample (one case in which the difference of two whole numbers is not a whole number) is sufficient to show that the set of whole numbers is not closed under subtraction.

C. Commutative property of addition

1. Have pupils compute the sums in each pair of examples:

   a. 14+17 17+14
   b. 36+52 52+36
   c. 69+138 138+69
   d. 4360+5955 5955+4360

2. Have them conclude that

   a. 14+17=17+14
   b. 36+52=52+36
   c. 69+138=138+69
   d. 4360+5955=5955+4360

3. Replace each □ with a numeral so that a true statement results.

   a. 25+19 = 19+□
   b. 38+22 = □ +38
   c. □+85 = 85+116
   d. 666+ □ = 222+666
   e. □+0 = 0+5

4. In 14+17=31 and 17+14=31, we say that the order of the addends is different. Does the order of the addends affect the sum?

5. Tell pupils that the property of addition which says that the order in which we add two whole numbers does not affect the sum is called the Commutative Property of Addition.

6. Refer to challenge.
   Have pupils see that they use the commutative property of addition in checking the addition of two whole numbers as follows:

$$\begin{array}{r} 59 \\ +86 \\ \hline 145 \end{array} \downarrow \begin{array}{l} \text{Add} \\ \text{downward} \end{array} \qquad\qquad \begin{array}{r} 59 \\ +86 \\ \hline 145 \end{array} \uparrow \begin{array}{l} \text{Add} \\ \text{upward} \end{array}$$

D. Subtraction is not commutative.

1. Consider: $8-5=?$
   $5-8=?$

2. Elicit that 8-5 is an element of the set of whole numbers. This is not so for 5-8. The differences are not the same. The order of the numbers does affect the result.

3. Have pupils conclude that subtraction is not commutative.

E. Zero is the identity element of addition.

1. Have pupils consider which replacement for each frame will result in a true statement.

$3 + \square = 3$ $\qquad$ $\square + 55 = 55$

$27 + \square = 27$ $\qquad$ $\square + 256 = 256$

2. Elicit that when each frame is replaced by 0, a true statement results.

$3 + 0 = 3$ - true $\qquad$ $0 + 55 = 55$ - true

$27 + 0 = 27$ - true $\qquad$ $0 + 256 = 256$ - true

3. Tell pupils that because the sum of zero and any number is the identical number, zero is called the identity element for addition.

4. Have pupils give statements showing commutativity for each of the following:

   a. $0 + 1 = 1$ $\quad$ $(1 + 0 = 1)$
   b. $0 + 13 = 13$
   c. $49 + 0 = 49$
   d. $2964 + 0 = 2964$

5. Is zero an identity element for subtraction? Is $84 - 0 = 84$?

NOTE: Zero is a right identity element only, because $84 - 0 = 84$, but $0 - 84$ is not equal to 84.

## II. Practice

A. Which of the following sets is closed under addition?

1. $\{0,1\}$ (No)
2. $\{0,1,2,3,4,5,8\}$ (No)
3. $\{5,10,15,20,\ldots\}$ (Yes)
4. $\{5,10,15,20\}$
5. $\{7,14,21,28,\ldots\}$

B. Is the set of whole numbers between 30 and 40 closed under addition? Explain.

C. Is the set of even numbers closed under addition? Explain.

D. Is the set of odd numbers closed under addition? Explain.

E. Is any set in A closed under subtraction?

F. Which of the following are commutative, that is, in which does the order not affect the result?

1. Putting on one's left glove; putting on one's right glove.
2. Putting on one's socks; putting on one's shoes.
3. Preparing lunch; eating lunch.
4. Preparing English homework; preparing math homework.

G. Which of the following are examples of the commutative property of addition?

1. $32 + 14 = 14 + 32$
2. $4 + 1 = 0 + 5$ NOTE: The sum is the same, but the addends are not. Therefore, this is not an illustration of the commutative property of addition.
3. $2 + 678 = 676 + 4$
4. $1 + 0 = 0 + 1$
5. $32678 + 1682 - 1682 = 32678$
6. $4 + 3 = 7$

H. Which of the following statements are true? For each true statement, state whether the commutative property or the identity element tells you that it is true.

a. 14 + 8 = 8 + 14

b. 72 + 29 = 29 + 72

c. 25 + 0 = 25

d. 98 + 119 = 119 + 98

e. 0 + 62 = 62

I. Add and check:

1. $\begin{array}{r}27\\ \underline{+38}\end{array}$

2. $\begin{array}{r}416\\ \underline{+\ 89}\end{array}$

3. $\begin{array}{r}6720\\ \underline{+4792}\end{array}$

4. $\begin{array}{r}47{,}112\\ \underline{+\ 9{,}014}\end{array}$

III. Summary

A. What do we mean when we say a set of numbers is closed under addition?

B. Name two sets of numbers that are closed under the operations of addition.

C. What name is given to the property of numbers which states that the order of adding two numbers does not affect the sum?

D. Name an operation under which the set of whole numbers is not closed.

E. Name an operation of whole numbers which is not commutative.

F. What is the identity element of addition? Explain.

G. What new vocabulary did you learn today?

(closure, commutative property of addition of whole numbers, identity element for addition)

## Lesson 15

**Topic: Addition of Whole Numbers**

**Aim: To understand and use the associative property of addition of whole numbers**

**Specific Objectives:**

Addition is a binary operation
Regrouping addends in an addition example does not change the sum: the associative property of addition
Subtraction is not associative
Regrouping can simplify addition

Motivation: What is the easiest way to compute the sum: 16 + 25 + 75?

I. Procedure

A. Binary operation

1. Have pupils add these numbers.

$$\begin{array}{r} 6 \\ 3 \\ \underline{7} \end{array}$$

2. Have them note that they added only two numbers at a time: 6+3, 9+7.
   Have them observe this as they compute other sums.

3. Tell pupils that addition is called a binary operation; an operation performed on two numbers at a time.

4. Have pupils note further that they grouped 6 and 3 to compute the sum, 9, and then added 7 to obtain a total of 16.

5. Recall that we can show this grouping by using parentheses: (6+3) + 7.

B. The associative property of addition of whole numbers

1. Elicit that in computing the sum 6+3+7, we may also group 3 and 7, compute the sum, 10, and add it to 6 to obtain a total of 16.

   a. How can we show this grouping? 6 + (3+7)
   b. Do (6+3) + 7 and 6 + (3+7) name the same number?
   c. Is (6+3) + 7 = 6 + (3+7) a true statement?

2. Consider the motivation example.

   a. Ask pupils to compute the sum.

   b. Some pupils will group 16 and 25, compute the sum, 41, and add 75 to the result, obtaining a total of 116.

      This can be expressed as $(16+25) + 75$.

      Some will group 25 and 75 and add 16 to the sum, 100, obtaining a total of 116.

      This can be expressed as $16 + (25+75)$.

      Pupils will notice that the second grouping will make the computation easier.

   c. Have pupils see that: $(16+25) + 75 = 16 + (25+75)$.

3. Have pupils find the sum of each of the following in two ways:

   a. $2 + 12 + 16$ c. $41 + 59 + 116$
   b. $19 + 15 + 22$ d. $205 + 25 + 75$

4. Have pupils conclude that the sum of three whole numbers is the same whether the first two addends are grouped or the last two addends are grouped. This is called the Associative Property of Addition.

5. How can we compute the sum of four (or more) whole numbers?

   a. Add: $2 + 7 + 6 + 8$

      1) We could think: $2 + 7 = 9$ (first sum)
         $9 + 6 = 15$ (second sum)
         $15 + 8 = 23$ (third sum)

      2) or, we could think: $2 + 7 = 9$ (first sum)
         $6 + 8 = 14$ (second sum)
         $9 + 14 = 23$ (third sum)

   b. Elicit that in addition of whole numbers, addends may be grouped in any way without changing the sum.

C. Subtraction is not associative

1. Have pupils consider whether there is an associative property of subtraction.

   Does $(11-6) - 4 = 11 - (6-4)$? Does $5 - 4 = 11 - 2$?
   $5 - 4 = 1$ and $11 - 2 = 9$
   $1 \neq 9$ (the symbol $\neq$ means "is not equal to")

2. Have pupils conclude that subtraction is not associative.

D. Simplifying addition by regrouping

1. Which sum is easier to compute without using pencil and paper?

   a. (25+75) + 87 or 25 + (75+87)

   b. (69+97) + 3 or 69 + (97 + 3)

2. Use the method you think easier to find the sums.

3. To add 56 and 32, we think: 56 + 30 is 86, and 2 more is 88. We have used: 56 + 32 = 56 + (30+2), which has been regrouped as: (56+30) + 2.

4. Compute these sums mentally.

   a. 25 + 62
   b. 17 + 22
   c. 71 + 27
   d. 63 + 55

5. Have pupils conclude that addition can often be simplified by suitable grouping.

II. Practice

A. Show that each of the following statements is true.

1. (18+2) + 15 = 18 + (2+15)
   20 + 15 = 18 + 17
   35 = 35

2. (29+79) + 1 = 29 + (79+1)

B. How can you tell without computing that the following statements are true?

1. (52+63) + 37 = 52 + (63+37)

2. 125 + (75+87) = (125+75) + 87

C. Replace each frame so that a true statement results.

1. 4 + (3+6) = (4 + □) + 6

2. (19+22) + 15 = □ + (22+15)

3. (38+ □) + 25 = 38 + (16+25)

4. □ + (29+8) = (63+29) + 8

D. Find the sum of each of the following in two ways.

1. 36 + 47 + 10
2. 89 + 22 + 51
3. 106 + 41 + 59
4. 71 + 29 + 250
5. 6 + 10 + 8 + 4
6. 8 + 5 + 3 + 9 + 6

E. All of the following statements are true.
Decide whether it is easier to compute the sum as given at the left of the equal sign, or as given at the right.
Then find the sum.

1. $(2+98) + 47 = 2 + (98+47)$
2. $60 + (40+73) = (60+40) + 73$
3. $(97+16) + 4 = 97 + (16+4)$
4. $(4+196) + 217 = 4 + (196+217)$

III. Summary

A. What is meant by the statement: Addition is a binary operation?

B. What name is given to the property of whole numbers which permits us to regroup addends?

C. Name an operation of whole numbers which is not associative?

D. Why do we often regroup addends in addition?

E. What new vocabulary did you learn today?

(binary operation; associative property of addition)

## Lessons 16 and 17

**Topic: Addition and Subtraction of Whole Numbers**

**Aim: To understand how our numeration system and the properties of addition are used in computation**

**Specific Objectives:**

How the properties of addition are used to compute sums .
Understanding the short forms for finding sums and differences
Introducing some ideas of proof

Challenge: How would you compute the following sum without using pencil and paper?

25 + 58 + 75

I. Procedure

A. Using the properties of addition in computation

1. Refer to challenge.
   Elicit that 25+ 58 + 75 may be grouped as (25+58) + 75 or as 25 + (58+75).

   a. Elicit that the easiest way to compute the sum would be to add 25 and 75 first, and then add 58 to this sum.

   b. How can we justify doing the computation in this way?

2. Have pupils justify each step of the computation as follows:

| | Reason |
|---|---|
| (25+58)+75 = 25+(58+75) | associative property |
| = 25+(75+58) | commutative property |
| = (25+75)+58 | associative property |
| = 100+58 | sum 25+75 |
| = 158 | sum 100+58 |

3. How are the properties of addition used in computing a sum such as 64+23?

| Sum 64+23 | Reason |
|---|---|
| 64+23 = (60+4)+(20+3) | expanded numeral |
| = 60+(4+20)+3 | addends may be grouped in any way |
| = 60+(20+4)+3 | order of addends may be changed |
| = (60+20)+(4+3) | addends may be grouped in any way |
| = 80+7 | sum 60+20; 4+3 |
| = 87 | sum 80+7 |

4. Have pupils supply the reasons for the steps in finding sums such as: $34+52$; $66+13$, and so on.

B. Understanding the short form for finding sums and differences

1. Have pupils supply the reasons in finding the sum: $27+61$.

| | Reasons |
|---|---|
| $27+61 = (20+7)+(60+1)$ | ? |
| $= 20+(7+60)+1$ | ? |
| $= 20+(60+7)+1$ | ? |
| $= (20+60)+(7+1)$ | ? |
| $= 80+8$ | ? |
| $= 88$ | ? |

2. The sum $27+61$ may be found by using a column form and renaming addends.

$$\begin{array}{rl} \underline{\text{Column}} & \underline{\text{Form}} \\ 27 & 20+7 \\ +61 & \underline{60+1} \\ & 80+8 = 88 \end{array}$$

a. Have pupils explain the column form in terms of the steps and reasons in 1.

b. Elicit that the form in which we usually show our addition is a short form of the above column form.

c. Have them do the addition using the short form.

3. Consider the following addition example:

$$\begin{array}{rl} 46 & 40+6 \\ \underline{+25} & \underline{20+5} \\ & 60+11 = 60+(10+1) \\ & \phantom{60+11} = (60+10)+1 \\ & \phantom{60+11} = 70+1 \\ & \phantom{60+11} = 71 \end{array}$$

a. How did we express 11 (the sum of the ones)?

b. What do we do when the sum of the ones is ten or greater?

c. Why is $60+(10+1) = (60+10)+1$ a true statement?

d. Have pupils do the addition using the short form.

4. Consider the following addition example:

$$\begin{array}{r} 353 \\ +\underline{274} \end{array} \qquad \begin{array}{rl} 300+50+3 & \\ \underline{200+70+4} & \\ 500+120+7 & = 500+(100+20)+7 \\ & = (500+100)+20+7 \\ & = 600+20+7 \\ & = 627 \end{array}$$

a. How did we express 120 (the sum of the tens)?

b. What do we do when the sum of the tens is ten tens or greater?

c. Why is $500+(100+20) = (500+100)+20$ a true statement?

d. Have pupils do the addition using the short form.

5. Consider the following subtraction example?

$$\begin{array}{r} 387 \\ -\underline{153} \end{array}$$

a. Have pupils express the numbers in expanded form and then subtract.

$$\begin{array}{r} 387 \\ \underline{-153} \end{array} \qquad \begin{array}{l} 300+80+7 \\ \underline{100+50+3} \\ 200+30+4 = 234 \end{array}$$

b. Use the expanded form and subtract as in a above.

$$1)\ \begin{array}{r} 2746 \\ -\underline{1632} \end{array} \qquad 2)\ \begin{array}{r} 4857 \\ -\underline{\ 145} \end{array} \qquad 3)\ \begin{array}{r} 36978 \\ -\underline{27657} \end{array}$$

6. Consider the following subtraction example:

$$\begin{array}{r} 84 \\ -\underline{36} \end{array} \qquad \begin{array}{l} 80+4 \\ \underline{30+6} \end{array} \qquad \begin{array}{l} 70+14 \\ \underline{30+6} \\ 40+8 = 48 \end{array} \qquad \begin{array}{rr} & 7\ \ 14 \\ & \cancel{8}\ \ \cancel{4} \\ - & \underline{3\ \ 6} \\ & 4\ \ 8 \end{array}$$

a. Why was 80+4 regrouped as 70+14?

b. Explain the short form as shown at the right.

7. Have pupils consider how 266 is subtracted from 528.

$$\begin{array}{r} 528 \\ -\underline{266} \end{array} \qquad \begin{array}{r} 500+20+8 \\ \underline{200+60+6} \end{array} \qquad \begin{array}{l} 400+120+8 \\ \underline{200+60\ +6} \\ 200+60\ +2 = 262 \end{array} \qquad \begin{array}{rrr} \overset{4}{\cancel{5}} & \overset{12}{\cancel{2}} & 8 \\ -2 & 6 & 6 \\ \hline 2 & 6 & 2 \end{array}$$

a. How was 528 finally renamed? Why?

b. Explain the short form.

8. Subtract, using first the expanded form and then the short form.

a. $\begin{array}{r} 34 \\ -\underline{27} \end{array}$ b. $\begin{array}{r} 81 \\ -\underline{69} \end{array}$ c. $\begin{array}{r} 795 \\ -\underline{467} \end{array}$ d. $\begin{array}{r} 200 \\ -\underline{183} \end{array}$

C. Some ideas of "proof"

1. Without using addition facts, how can you prove that (3+4)+5 = 3+(5+4)?

a. Tell pupils that in mathematics we are often interested in proving statements. We do this by writing a series of steps and giving as a reason for each, basic properties we have previously agreed upon.

b. To prove (3+4)+5 = 4+(5+3)

| Statements | Reasons |
|---|---|
| (3+4)+5 = (4+3)+5 | commutative property |
| = 4+(3+5) | associative property |
| = 4+(5+3) | commutative property |

II. Practice

A. Find these sums mentally.

1. 42+36
2. 23+54
3. 72+25
4. 66+13
5. 6+18+94
6. 98+112+2
7. 75+315+25
8. 8+109+192

B. Replace the frames.

$$\begin{aligned} (30+5) + (20+3) &= 30 + (\square + 20) + 3 \\ &= 30 + (\triangle + 5) + 3 \\ &= (30 + \triangle) + (5 + 3) \\ &= \diamond + 8 \\ &= 58 \end{aligned}$$

C. Replace the frames

$$\begin{array}{rcl}
\begin{array}{r} 56 \\ \underline{+37} \end{array} & = & \begin{array}{l} 50 + 6 \\ \underline{30 + 7} \\ 80 + (\square + 3) \end{array} \\
 & = & (80 + \square) + 3 \\
 & = & \triangle \quad + 3 \\
 & = & \diamond
\end{array}$$

D. Find the sums first by naming the addends in expanded form, and then by using the short form.

$$1.\ \begin{array}{r} 215 \\ \underline{+109} \end{array} \qquad 2.\ \begin{array}{r} 128 \\ \underline{+397} \end{array} \qquad 3.\ \begin{array}{r} 4260 \\ \underline{+\ 723} \end{array} \qquad 4.\ \begin{array}{r} 6012 \\ \underline{+4166} \end{array}$$

E. Use the short form to find the sums.

$$1.\ \begin{array}{r} 138 \\ 402 \\ \underline{665} \\ \phantom{0} \end{array} \qquad 2.\ \begin{array}{r} 4318 \\ 176 \\ \underline{9051} \\ \phantom{0} \end{array} \qquad 3.\ \begin{array}{r} 2215 \\ 1468 \\ 5813 \\ \underline{6961} \end{array} \qquad 4.\ \begin{array}{r} 613 \\ 7819 \\ 2344 \\ \underline{8001} \end{array}$$

F. Give the reason in place of the question mark.

$$\begin{array}{rll}
63 + 35 & = (60+3) + (30+5) & ? \\
 & = 60 + (3+30)+5 & ? \\
 & = 60 + (30+3)+5 & ? \\
 & = (60+30) + (3+5) & ? \\
 & = 90 + 8 & \\
 & = 98 &
\end{array}$$

G. Subtract

$$1.\ \begin{array}{r} 49 \\ \underline{-16} \end{array} \qquad 2.\ \begin{array}{r} 76 \\ \underline{-59} \end{array} \qquad 3.\ \begin{array}{r} 1248 \\ \underline{-\ 376} \end{array} \qquad 4.\ \begin{array}{r} 5820 \\ \underline{-1966} \end{array} \qquad 5.\ \begin{array}{r} 87{,}092 \\ \underline{-\ 9{,}008} \end{array}$$

H. Prove the following statements:

1. $(17+30)+13 = 30+(17+13)$
2. $(25+19)+75 = (25+75)+19$

III. Summary

A. In computing sums, what properties of addition do we often make use of?
B. What is meant when we say we will "prove" a statement of equality?
C. In proving statements of equality, what properties of addition do we often make use of?

## Lesson 18

Topic: Addition and Subtraction of Whole Numbers

Aim: To learn to solve problems involving addition and subtraction of whole numbers

NOTE: The difficulties that pupils often encounter in attempting to solve verbal problems may be traced to inadequate understanding of the meaning of fundamental operations.

Although the problem situations which are presented in the following procedures are simple, they are being included in order to help pupils gain insight into the meaning of the operations of addition and subtraction and their application.

Specific Objectives:

- To reinforce the meaning of the operations of addition and subtraction
- To develop steps in analyzing a problem:
  - determine what is to be found
  - determine the facts given
  - decide upon the operation(s) which will relate the given facts to what is to be found
  - eliminate from consideration irrelevant information and recognize when there is insufficient information
- To recognize that a problem may often be solved in more than one way

Motivation: In solving a word problem, how can you tell which operation(s) - addition, subtraction, multiplication, division - should be used?

I. Procedure

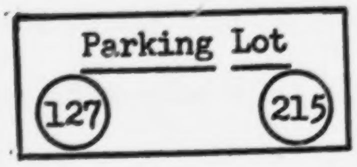
A. Reinforce the meaning of addition and of subtraction

1. The meaning of addition

a. Consider example:

There are 127 cars in one part of a parking lot, and 215 in another part of the lot. How many cars are there in the parking lot?

b. Diagram the situation.



c. Have pupil recognize that there are two disjoint sets: one consisting of 127 elements, and the other of 215 elements. Pupils see that the number of elements in the union of the two sets is the number of cars in the parking lot. The problem is to find the number of elements in the union.

What arithmetic operation is used to find the total of the numbers of the elements of the two sets? (addition)

d. How can we express the relationship in a mathematical sentence?

$$127 + 215 = \square$$

We can use the column form.

$$\begin{array}{r} 127 \\ +215 \\ \hline \square \end{array}$$

The number of cars in the parking lot is 342.

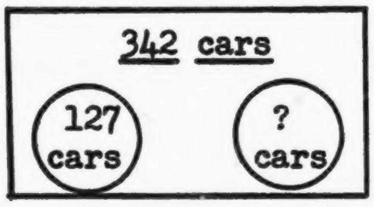
e. Have pupils see that addition is used to find the total of the numbers of two or more disjoint sets. (Motivation)

2. The meaning of subtraction

a. Consider example:

There are 342 cars in two sections of a parking lot. In one section of the parking lot there are 127 cars. How many cars are there in the other section?

b. Diagram the situation.



c. Have pupils realize that they can consider all the cars in the lot as the union of the set of cars in one section, and the set of cars in the other section.

d. The number of the set of cars in one section is 127. If we add the number of the set of cars in the other section to 127, we have the number of the set which is their union (342).

How would you express this as a mathematical sentence?

$$127 + \square = 342$$

e. What arithmetic operation is used to find the "missing addend"? (subtraction)
How would you express this as a mathematical sentence?

$342 - 127 = \square$

We can use the column form.

$$\begin{array}{r} 342 \\ -\underline{127} \\ \square \end{array}$$

The number of cars in the other section is 215.

f. Using this and similar procedures, have pupils see that

1) if the total of the numbers of two sets is known, and if the size of one of the sets is known, we use subtraction to find the size of the other set (motivation)

2) if we wish to find the difference of the numbers of two sets, we use subtraction (motivation).

3. Have pupils state the operation(s) to be used in solving each of the following problems.

a. How many in all are in a set of 120 and a set of 85, if the two sets are disjoint?

b. From a set of 50, remove a set of 22.
How many are left?

c. How many more are there in a set of 276 than there are in a set of 198?

d. How many altogether are in a set of 48, a set of 206, and a set of 413, if these sets are disjoint?

B. Analyzing problems

1. Consider the following problem:

John's father, who is 40 years old, modernized the kitchen in his home. He bought a new stove for $168 and a new sink for $85. If he had $850 in his bank account, how much was left after paying for the new equipment?

a. What is to be found? (the number of dollars left in the bank account)

b. What facts are given? (the number of dollars in the cost of the stove; in the cost of the sink; in the bank account originally)

c. What operations will we use to relate the given facts to what we wish to find?

1) We must first find the total cost of the equipment. What operation shall we use to do this? (addition, since we are finding the total of the numbers of two disjoint sets, $168 and $85)

$$168 + 85 = \square \quad \text{or} \quad \begin{array}{r} 168 \\ \underline{+\ 85} \\ \square \end{array}$$

The total cost of the equipment is $253.

2) We then find the amount left in the bank account after the new equipment was paid for. What operation is needed for this? (subtraction, since we are finding the difference of the numbers of two sets, $850 and $253)

$$850 - 253 = \square \quad \text{or} \quad \begin{array}{r} 850 \\ \underline{-253} \\ \square \end{array}$$

There was $597 left in the bank account.

d. What facts are irrelevant?

1) Did the man's age help you arrive at the answer?
2) Point out the importance of recognizing what facts, if any, are irrelevant.

2. Alternate method of solving the problem in 1.

a. Elicit that we could first subtract to find the amount left in the bank account after paying for the stove.

$$850 - 168 = \square \quad \text{or} \quad \begin{array}{r} 850 \\ \underline{-168} \\ \square \end{array}$$

The amount left is $682.

b. We could then subtract again to find the final amount left after paying for the sink.

$$682 - 85 = \square \quad \text{or} \quad \begin{array}{r} 682 \\ \underline{-\ 85} \\ \square \end{array}$$

The final amount left is $597.

3. Consider the problem:

Jane is three years younger than her sister. How old is Jane?

a. What are we to find? (the number of years in Jane's age)
b. What facts are given? (the relationship between Jane's age and her sister's age)
c. Elicit that Jane's sister's age is not known. We cannot solve this problem because there is insufficient information provided.

## II. Practice

Some of the following problems cannot be solved until some additional fact has been supplied. Solve as many of the problems as you can. Tell what additional fact is needed for each of the other problems.

A. A plane flies 165 miles from Tallahassee to Jacksonville. It then travels 768 miles from Jacksonville to Washington. How far does it travel by this route?

B. Harry scored 135 in a bowling game. How much more did Harry score than Ira?

C. How much change will Sue have from a $5 bill after she spends $2.67 at the supermarket and $1.24 at the bakery?

D. At a sale, a pair of shoes was sold for $9.50 and a suit was sold for $42.50. How much would you save by buying during the sale?

E. One year, on national forest land, 8235 fires were caused by lightning, 1046 by campers, 1570 by smokers, and 2014 by other causes. How many fires were caused by campers and smokers? by lightning and other causes?

F. Mt. Whitney in California is 14,495 feet high. The members of a scientific expedition camped at a spot about 8,500 feet up. About how far from the top of Mt. Whitney was the camp situated?

G. Ann took two trips one year. She traveled 459 miles on one trip and 923 miles on the other. Marcia took a trip of 2481 miles. Janet traveled 1502 miles. Who traveled the farthest, Ann, Marcia, or Janet? How much farther did she travel than each of the others?

H. Mrs. Pedros bought a pair of curtains for her kitchen window for $5.25 and a tablecloth for $3.98. She had thought of buying a tablecloth for $4.50. How much did she pay altogether for her purchases?

1. What information is irrelevant in this problem?
2. Formulate another question that might have been asked about the problem.
3. Find the answer to the question you formulated.

I. Consult textbooks for additional problems.

## III. Summary

A. Elicit the steps in analyzing a problem.
   1. Determine what is to be found.
   2. Determine what information is given.
   3. Decide upon the operations to be used.

B. How do you decide on the operation to be used?

C. Which of the steps will show you whether you have too much information? too little information?

D. What does the fact that your teacher says, "Who did this another way"? tell you about solving problems?

## Lesson 19

Topic: Multiplication of Whole Numbers

Aim: To understand how sets may be used to illustrate multiplication of whole numbers

Specific Objectives:

Multiplication in terms of the union of equivalent disjoint sets
Multiplication in terms of a product set

Motivation: If you were permitted to use only the process of addition, how long would it take to compute the number of gallons of water used by an electric power plant in one year if it uses 2,350,000 gallons every 24 hours?

I. Procedure

A. Multiplication in terms of the union of equivalent disjoint sets

1. Have pupils realize that the computation for the motivation problem would take a long time because we would have to find the sum of 365 addends. To do such a computation quickly and skillfully, we make use of multiplication facts.

2. Consider the example:

There are 36 pupils registered in each of four classes. How many pupils are there in all four classes?

a. Diagram the situation.

$$\boxed{36} + \boxed{36} + \boxed{36} + \boxed{36} = \square$$

b. Have pupils recognize that there are four equivalent disjoint sets. (36 elements in each set) The number of the union of the four sets is the number of pupils on register.

What arithmetic operation is used to find the number of the union of the four sets? (addition)

c. What is a shorter way of finding the number of elements in the union when the sets are equivalent? (multiplication) How do we express this as a mathematical sentence?

$$4 \times 36 = \square \quad \text{or} \quad \begin{array}{r} 36 \\ \times\ \underline{4} \\ \square \end{array}$$

There are 144 pupils in all four classes.

d. Have pupils see that multiplication is a shorter way of finding the total of the numbers of two or more equivalent disjoint sets.

B. Multiplication as a product set

1. Consider the following two sets:

   B = {Charles, Bob, Tom}      G = {Eve, Carla}

   Elicit that the sets are disjoint.

2. Have pupils pair each member of set B with each member of set G, as, for example,

   Charles-Eve, Charles-Carla, and so on.

   a. How many such pairs can you find if the first member of each pair is taken from set B and the second member of each pair is taken from set G?

   b. Tell pupils that the set of all pairs made up from elements in set B and in set G is called the product set of set B and set G.

   Product set of set B and set G

   |  |  |  |
   |---|---|---|
   | Charles-Eve | Bob-Eve | Tom-Eve |
   | Charles-Carla | Bob-Carla | Tom-Carla |

   c. How many elements are in set B? (3)
   How many elements are in set G? (2)
   What is 3 x 2?
   How many elements are in the product set?

3. Have pupils name all possible pairs made up by pairing members of set L with members of set N.

   L = {A,B,C,D,E}
   N = {1,2,3}

   a. How many elements are in set L?  in set N?
   b. How many elements are in the product set of set L and set N?
   c. How can we find the number of elements in the product set without forming pairs?

4. Have pupils conclude that the pairing of each member of one set with each member of another set may be used to compute products.

II. Practice

A. How many elements are in the product set of the following sets?

$A = \{\text{hat, gloves, shoes}\}$

$B = \{\text{sweater, skirt, blouse}\}$

B. Check your answer to A by listing all the pairs in the product set.

C. A boy has three jackets and six ties. What is the total number of jacket-tie combinations he can make?

D. A girl has a set of five dresses but only one hat. How many hat-dress combinations are there in all?

E. A girl has a set of five dresses, but no hats.

1. What is the name of the set of hats?
2. What is the number of the set of hats?
3. What mathematical sentence describes the number of hat-dress combinations? ($0 \times 5 = \square$)

F. Compute:

1. $4 \times 5 = \square$
2. $3 \times 7 = \square$
3. $8 \times 6 = \square$
4. $6 \times 8 = \square$
5. $1 \times 9 = \square$
6. $0 \times 12 = \square$
7. $9 \times 5 = \square$
8. $2 \times 14 = \square$
9. $5 \times 0 = \square$

III. Summary

A. What is the meaning of $3 \times 5$? ($5+5+5$) of $5 \times 3$? ($3+3+3+3+3$)

B. Why is the multiplication of whole numbers often referred to as a short form of addition?

C. How is the number of elements in the union of five equivalent disjoint sets related to the number of elements in each of the given sets?

D. What is meant by the product set of two sets?

E. If you know the number of elements in each of two sets, how do you find the number of elements in their product set?

F. What new vocabulary have you learned today? (product set)

## Lesson 20

Topic: Properties of Multiplication of Whole Numbers

Aim: To understand the closure and commutative properties of multiplication

Specific Objectives:

To understand:
- Closure for multiplication of whole numbers
- The commutative property of multiplication
- The identity element of multiplication
- The multiplicative property of zero

Motivation: When we compute the product in a multiplication example, why do we often check our work by interchanging the factors?

I. Procedure

A. The closure property of multiplication of *whole numbers*

1. Review the closure property of addition of whole numbers. Is there a similar property for the operation of multiplication?

2. Consider the set: {0,1,2,3,4,...}.

    a. What is the product of the factors 4 and 3? Is the product a whole number?

    b. What is the product of 500 and 700? Is the product a whole number?

    c. Does it seem that for any two whole numbers there is a product which is also a whole number?

3. Tell pupils that since the product of two whole numbers is always a whole number, we say that the set of whole numbers is closed under the operation of multiplication. This is called the **Closure Property of Multiplication for Whole Numbers**.

4. Is the set of whole numbers closed under division?

    a. Consider the set: {0,1,2,3,4,...}.

        1) Is the result of dividing two whole numbers always a whole number?

$35 \div 5 = 7 \qquad 0 \div 3 = 0$

$6 \div 6 = 1 \qquad 8 \div 5 = ?$

2) Have pupils conclude that the result of dividing two whole numbers is not always a whole number.

b. Elicit that the set of whole numbers is not closed under division.

B. Commutative property of multiplication

1. Given the example: $\begin{array}{r} 7 \\ \underline{\times 48} \end{array}$ we would probably change it to $\begin{array}{r} 48 \\ \underline{\times 7} \end{array}$.

Do we obtain the same product?

2. Recall the check for multiplication - interchange the order of the factors. What result do you expect?

$$\begin{array}{r} 68 \\ \underline{\times 79} \\ 612 \\ \underline{476\phantom{0}} \\ 5372 \end{array} \qquad\qquad \begin{array}{r} 79 \\ \underline{\times 68} \\ 632 \\ \underline{474\phantom{0}} \\ 5372 \end{array}$$

3. Have pupils conclude that the order of two factors in multiplication does not affect the product. This is called the Commutative Property of Multiplication for Whole Numbers.

4. Is division of whole numbers commutative?

a. Consider:

$20 \div 5 = 4$

$5 \div 20 = \frac{1}{4}$

$4 \neq \frac{1}{4}$

Is $20 \div 5 = 5 \div 20$ true?

b. Elicit that since the quotient is not the same when dividend and divisor are interchanged, division of whole numbers is not commutative.

C. The identity element of multiplication

1. Have pupils study the following pattern:

1 x 0 = 0
1 x 1 = 1
1 x 2 = 2

1 x 3 = 3
1 x 4 = 4
1 x 5 = 5

1 x 6 = 6
1 x 7 = 7
1 x 8 = 8, and so on

2. Have them use the pattern to suggest replacements for the frames which will result in true statements.

$1 \times 5 = \square$
$1 \times 9 = \square$
$25 \times \square = 25$
$1 \times \square = 39$

3. Elicit that the product of one and a number, or a number and one is the number. Therefore, one is said to be the identity element for multiplication.

4. Is one an identity element for division?

   a. Find the quotients

   $10 \div 1 = \square$

   $23 \div 1 = \square$

   $100 \div 1 = \square$

   $0 \div 1 = \square$

   b. Elicit that the quotient of any whole number and 1 is the number. Therefore, one is the identity element for division when 1 is the divisor. ($1 \div 10 \neq 10$)

D. The multiplicative property of zero

1. Have pupils study the following pattern:

$0 \times 0 = 0$
$0 \times 1 = 0$
$0 \times 2 = 0$
$0 \times 7 = 0$

$0 \times 10 = 0$
$0 \times 25 = 0$
$0 \times 100 = 0$
$0 \times 250 = 0$, and so on

2. Have them use the pattern to suggest replacements for the frames which will result in true statements.

$0 \times 3 = \square$
$0 \times 4 = \square$
$\square \times 87 = 0$
$0 \times \square = 0$

3. Elicit that the product of zero and a number is zero.

4. What is the quotient of a number by zero? (Division by zero is meaningless.)

## II. Practice

A. Which of these sets is closed under multiplication?

1. {0,1,2,3,4,5} (No)
2. {0,1} (Yes)
3. {2,4,6,8}
4. {2,4,6,8,...}
5. {5,10,15,20,...}

B. Is the set of odd numbers closed under multiplication? Explain.

C. Is any set in A closed under division?

D. Which of the following are illustrations of the commutative property of multiplication?

1. 4 x 8 = 8 x 4
2. 5 x 12 = 30 x 2
3. 6 x 9 = 9 x 6
4. 10 x 18 = 18 x 10
5. 5 x 20 = 10 x 10

E. Replace each frame so that a true statement results.

1. 7 x 8 = 8 x □
2. 12 x 14 = □ x 12
3. □ x 25 = 25 x 4
4. 8 x □ = 16 x 8
5. 15 x 15 = 15 x □

F. How do these arrangements illustrate the commutative property of multiplication?

```
xxx            xxxxx
xxx            xxxxx
xxx            xxxxx
xxx
xxx
```

G. Multiply and then apply the commutative property of multiplication in checking each of the following:

1. $\begin{array}{r}36\\ \underline{\times 57}\end{array}$
2. $\begin{array}{r}84\\ \underline{\times 97}\end{array}$
3. $\begin{array}{r}76\\ \underline{\times 69}\end{array}$

H. Which property of multiplication does each of the following illustrate?

1. 2 x 7 = 7 x 2
2. 1 x 68 = 68
3. 9 x 0 = 0
4. 1 x 1 = 1
5. 32 x 64 = 64 x 32

I. Which of the following are names for zero? names for one?

1. $0 + 1$
2. $9 \times 0$
3. $\frac{6 \times 1}{6}$
4. $\frac{4 \times 5}{20}$
5. $9 \times 3 \times 7 \times 0$
6. $\frac{2^3}{2^3}$
7. $\frac{2-2}{1}$
8. $1 \times \frac{3}{3}$
9. $1 + (1-1)$
10. $0 \times 26 \times 87$

J. OPTIONAL

Consider the following new operation. It is defined in terms of familiar operations.

○ means: Add the second number to twice the first number.

For example, $3 \bigcirc 4 = (2 \times 3) + 4 = 6 + 4 = 10$

1. Compute the answers.

a. $5 \bigcirc 3$
b. $3 \bigcirc 5$
c. $6 \bigcirc 11$
d. $11 \bigcirc 6$
e. $13 \bigcirc 15$
f. $15 \bigcirc 13$
g. $12 \bigcirc 20$
h. $20 \bigcirc 12$

2. Examine the answers to pairs a and b, c and d, e and f, g and h. Is ○ a commutative operation? Explain.

III. Summary

A. What is meant by the closure property of multiplication of whole numbers?

B. What name is given to the property of numbers which states that the order of multiplying two numbers does not affect the product?

C. Name two operations of whole numbers which are not commutative.

D. What is the identity element for multiplication?

E. If one of several factors is zero, what is true of the product?

## Lesson 21

Topic: Properties of Multiplication of Whole Numbers

Aim: To understand and use the associative property of multiplication

Specific Objectives:

- Multiplication is a binary operation
- The associative property of multiplication
- Regrouping can simplify multiplication
- Using both the commutative and associative properties in computing products

Challenge: What is the easiest way of computing 12 x 4 x 25?

I. Procedure

A. Multiplication is a binary operation

1. Consider the problem:

   If a carton of eggs contains 12 eggs, what is the total number of eggs in 2 boxes, each of which contains 8 cartons of eggs?

   a. Elicit that to solve the problem, we could compute the the following product: 2 x 8 x 12.

      $2 \times 8 = 16$ $\qquad$ $16 \times 12 = 192$

      There are 192 eggs in the two boxes.

   b. Have pupils note that they multiplied only two numbers at a time. They first found the total number of cartons in both boxes (2x8 or 16), and then the total number of eggs in these cartons (16x12 or 192).

2. Elicit that in computing the product of three or more factors, we multiply only two numbers at a time. Multiplication is therefore a binary operation.

3. How could we use parentheses to show that we first computed the product of 2 and 8, and then multiplied 12 by this product? (2x8)x12.

B. The associative property of multiplication

1. In what other way can we arrive at the solution to the problem in A-1?

Elicit that we can first find the number of eggs in one box. We then find the total number of eggs in both boxes.

$8 \times 12 = 96$ $2 \times 96 = 192$

2. How would we use parentheses to show that this time we first computed the product of 8 and 12, and then multiplied this product by 2?

   $2\times(8\times12)$

3. Elicit that to find the product of three factors such as $2 \times 8 \times 12$, we usually think of grouping the factors as $(2\times8)\times12$. We may also group them $2\times(8\times12)$. The products are the same. $(2\times8)\times12 = 2\times(8\times12)$

4. Compute the products by grouping factors in two different ways. Compare your answers in each case.

   a. $3 \times 9 \times 5$
   b. $5 \times 8 \times 3$
   c. $2 \times 4 \times 8$
   d. $3 \times 5 \times 3$
   e. $5 \times 8 \times 2$
   f. $6 \times 2 \times 3$

5. Have pupils conclude that the product of three factors is the same whether the first two factors are grouped or the last two factors are grouped. This is called the Associative Property of Multiplication.

   For example: $(5\times7)\times10 = 5\times(7\times10)$.

6. Is division associative?

   a. Does $(8 \div 4) \div 2 = 8 \div (4 \div 2)$?
      Does $2 \div 2 = 8 \div 2$?
      $1 \neq 4$

   b. Pupils conclude that division is not associative.

C. Simplifying multiplication by the use of the associative property

   1. Have pupils compute each of these products in two ways.

      a. $12 \times 4 \times 25$ (challenge)

      | | |
      |---|---|
      | $(12\times4)\times25$ | $12\times(4\times25)$ |
      | $48 \times25$ | $12\times100$ |
      | $1200$ | $1200$ |

      b. $17 \times 50 \times 2$

      c. $15 \times 25 \times 4$

      d. $36 \times 5 \times 20$

Which grouping in each example makes the computation easier?

2. Use the associative property to find the answers easily.

   a. 5x(4x12) c. (7x17)x10
   b. (8x16)x5

3. To compute the product 3x30, we think:

|  | Reasons |
|---|---|
| 3x30 = 3x(3x10) | expanded numeral |
| = (3x3)x10 | associative property |
| = 9 x 10 |  |
| = 90 |  |

4. Find the products without using pencil and paper.

   a. 6 x 40 c. 5 x 6000
   b. 8 x 300 d. 9 x 7,000,000

D. Using both the commutative and associative properties in computing products

1. How could you compute the following product without using pencil and paper? 25 x 8 x 4

   a. Elicit that we may group the factors as (25x8)x4 or as 25x(8x4)

   b. Elicit that the easiest way to compute this product would be **first to** multiply 4 by 25 and then multiply 8 by this result.

   How can we justify doing the computation in this way? (25x4)x8

2. Have pupils justify each step of the computation as follows:

|  | Reason |
|---|---|
| (25x8)x4 = 25x(8x4) | associative property |
| = 25x(4x8) | commutative property |
| = (25x4)x8 | associative property |
| = 100x8 | product 25x4 |
| = 800 | product 100x8 |

## II. Practice

A. Compute the products by grouping factors in two different ways. Compare your answers in each case.

1. 60 x 5 x 9
2. 57 x 4 x 25
3. 18 x 21 x 5

B. Replace each frame to make a true statement.

1. (18x3)x2 = 18 x (3 x □)
2. (5x9)x1 = 5 x (□ x 1)
3. (15x6)x3 = □ x (6 x 3)
4. (9x8)x□ = 9x(8x12)
5. (□x10)x12 = 4x(10x12)

C. Illustrate the associative property using each of the following:

1. (17x2)x9
2. (67x38)x11
3. 4610x(817x368)
4. (5,675,193 x 3)x4128

D. Use the associative property to find the answers easily.

1. (63x2)x5
2. 4x(25x9)
3. 50x(2x18)
4. (12x6)x5

E. Replace the frames with either =, <, or > to make each of the following a true statement.

1. (12x8)x5 ▭ 12x(8x5)
2. (16÷4)÷2 ▭ 16÷(4÷2)
3. 18-(5-1) ▭ (18-5)-1
4. 14x(2x9) ▭ (14x2)x9

F. Use the commutative and associative properties to find the answers easily.

1. 2x33x50
2. 20x68x5
3. 35x6x2
4. 45x8x2

## III. Summary

A. What is meant by the statement, "Multiplication is a binary operation"?
B. What name is given to the property of whole numbers which permits us to regroup factors?
C. Which operations of whole numbers are not associative?
D. In simplifying multiplication, what properties of whole numbers do we make use of?

## Lessons 22 and 23

**Topic:** Properties of Whole Numbers

**Aim:** To understand and use the distributive property of multiplication over addition for whole numbers

**Specific Objectives:**

To learn that the operation of multiplication of whole numbers is distributive over addition
To have pupil realize that he has been using this distributive property in multiplying whole numbers

**Challenge:** The school G.O. store had a two-day sale of athletic shirts. The price of a shirt was \$2. The first day 15 shirts were sold; the second day 20 shirts were sold. What was the total amount of money taken in from the sale of the shirts?

I. Procedure

A. The distributive property of multiplication over addition

1. Elicit explanations of how pupils can arrive at the answer to the challenge problem.

   a. A pupil may say he added 15 and 20 and multiplied the sum by 2. This is recorded as:

      $2\times(15+20) = 2\times(35) = 70$

   b. Another pupil may say that he multiplied 15 by 2, and 20 by 2, and added the products. This is recorded as:

      $(2\times15) + (2\times20) = 30 + 40 = 70$

   c. Have them realize that both methods result in 70, or that

      $2\times(15+20) = (2\times15) + (2\times20)$

2. Compute the answers to each pair of exercises to see if they are equal. Show each step as in the example.

   a. $4\times(2+3)$; $(4\times2) + (4\times3)$

      $4\times(2+3) = 4\times5 = 20$

      $(4\times2) + (4\times3) = 8 + 12 = 20$

      Therefore, $4\times(2+3) = (4\times2) + (4\times3)$

b. $7\times(6+8)$; $(7\times6)+(7\times8)$

c. $5\times(12+8)$; $(5\times12)+(5\times8)$

d. $0\times(6+5)$; $(0\times6)+(0\times5)$

3. Tell pupils that each of the above exercises illustrates the Distributive Property of Multiplication Over Addition.

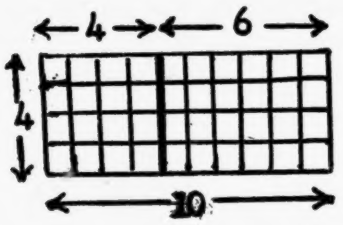
   a. How many operations are involved in this distributive property?

   b. What operations are involved in this distributive property?

4. The distributive property of multiplication over addition may be expressed as either a "right distributive property" or a "left distributive property."

   Thus, $2\times(3+4)$ is an example of a left distributive propery, and $(5+8)\times6$ is an example of a right distributive property.

5. Have pupils see how the distributive property of multiplication over addition is involved as they find the number of squares in the rectangle below in two ways.



   a. First way: $(4\times4)+(4\times6) = 16+24$
   $= 40$

   b. Second way: $4\times(4+6) = 4\times10$
   $= 40$

6. Replace the frames.

   a. $4\times(9+8) = (4\times9)+(4\times\Box)$

   b. $6\times(2+\Box) = (6\times2)+(6\times6)$

   c. $12\times(\Box+11) = (12\times8)+(12\times11)$

   d. $(\Box\times5)+(\Delta\times9) = (7+1)\times9$

   e. $(13+8)+(\Box\times10) = 13\times(8+10)$

B. Using the distributive property

1. Have pupils consider computing the product of a number greater than 10 by a number less than 10. (one-place multiplier)

   a. Compare the following methods of computing the product 3x23.

      1) First method

$$\begin{aligned} 3\times 23 &= 3\times(20+3) && \text{expanded numeral} \\ &= (3\times 20) + (3\times 3) && \text{distributive property} \\ &= 60+9 \\ &= 69 \end{aligned}$$

      2) Second method

$$\begin{array}{r} 20 \\ \times\ \underline{3} \\ 60 \end{array} \qquad \begin{array}{r} 3 \\ \times\underline{3} \\ 9 \end{array} \qquad \begin{array}{r} 60 \\ +\ \underline{9} \\ 69 \end{array}$$

      3) Third method

$$\begin{array}{r} 23 \\ \times\ \underline{3} \\ 69 \end{array}$$

   b. Have pupils realize that when we multiply a number greater than 10 by a number less than 10, we are making use of the distributive property.

   c. Have them see that the third (and usual) method is a short way of computing the products. In this method we do not show each step in detail, but retain some facts in our heads.

2. Computing products of factors named by 2 digits (2-place multiplier)

   a. Consider 35 x 42

      1) Have pupils recall the vertical form for multiplying two numbers greater than 10.

$$\begin{array}{rl} 42 & \\ \times\underline{35} & \\ 10 & \text{first partial product} \\ 200 & \text{second partial product} \\ 60 & \text{third partial product} \\ \underline{1200} & \text{fourth partial product} \\ 1470 & \text{product} \end{array}$$

2) Have them see that the distributive property explains why we first obtain partial products and then add the partial products to obtain the product.

$$\begin{aligned} 35\times42 &= 35\times(40+2) && \text{expanded numeral} \\ &= (35\times40) + (35\times2) && \text{distributive property} \\ &= (30+5)\times40 + (30+5)\times2 && \text{expanded numeral} \\ &= (30\times40) + (5\times40) + (30\times2) + (5\times2) && \text{distributive property} \\ &= 1200 + 200 + 60 + 10 && \text{partial products} \\ &= 1470 && \end{aligned}$$

Are the partial products the same as those named in the vertical form?

b. Have pupils compare the following two methods of computing 26x37.

__Vertical Form__

$$\begin{array}{r l} 32 & \\ \underline{\times26} & \\ 192 & \text{first partial product} \\ \underline{640} & \text{second partial product} \\ 832 & \text{product} \end{array}$$

__Expanded Form__

$$\begin{aligned} 26\times32 &= (20+6)\times32 \\ &= (20\times32) + (6\times32) \\ &= 640 + 192 \\ &= 832 \end{aligned}$$

1) In the vertical form, which two numbers are multiplied to obtain the first partial product? (32 and 6)
to obtain the second partial product? (32 and 20)

2) In the expanded form, which factor was expressed in expanded form?
In which step was the distributive property used?

3) Are the partial products named in the expanded form the same as those named in the vertical form?

3. Computing products of factors named by 3 digits (3-place multipliers)

a. Consider:

$$\begin{array}{r l} 325 & \\ \times\underline{127} & \\ 2275 & \text{first partial product} \\ 6500 & \text{second partial product} \\ \underline{32500} & \text{third partial product} \\ 41275 & \text{product} \end{array}$$

1) Which two numbers were multiplied to obtain the first partial product? the second partial product? the third partial product?

2) What must be done to the three partial products to obtain the product of 127 and 325?

3) What is the product?

b. Have pupils explain the multiplication of 127 and 325 in terms of the distributive property.

$$\begin{aligned} 127\times325 &= (100+20+7)\times325 && \text{expanded numeral}\\ &= (100\times325) + (20\times325) + (7\times325) && \text{distributive property}\\ &= \phantom{(}32500 \ + \ 6500 \ + \ 2275\\ &= \phantom{(}41275 \end{aligned}$$

c. The final algorithm is:

$$\begin{array}{r} 325\\ \times\underline{127}\\ \underline{2275}\\ 650\\ \underline{325\phantom{00}}\\ 41275 \end{array}$$

Have pupils realize:

1) In the second partial product, 650 means 650 tens because 325 has been multiplied by 2 tens, or 20.
2) In the third partial product, 325 means 325 hundreds because 325 has been multiplied by 1 hundred, or 100.

4. For pupils who have an inadequate understanding of the multiplication algorithm, the algorithm may be redeveloped as follows:

a. Review multiplication by 10 and multiples of 10; by 100 and multiples of 100.

b. Consider 127x325 or $\begin{array}{r} 325\\ \times\underline{127} \end{array}$

1) Estimate $100\times300 = 30{,}000$ $\quad 127\times325 > 30{,}000$

2) Multiply by hundreds first, then by tens, and then by ones.

$$\begin{array}{rcrcrcr} 325 & \qquad & 325 & \qquad & 325 & \qquad & 32500\\ \times\underline{100} & & \underline{\times\ 20} & & \underline{\times\ \ 7} & & 6500\\ 32500 & & 6500 & & 2275 & & \underline{2275}\\ & & & & & & 41275 \end{array}$$

3) We could multiply by ones first, then by tens, and then by hundreds.

$$\begin{array}{r}325\\ \underline{\times\quad 7}\\ 2275\end{array}\qquad\begin{array}{r}325\\ \underline{\times\ 20}\\ 6500\end{array}\qquad\begin{array}{r}325\\ \underline{\times 100}\\ 32500\end{array}\qquad\begin{array}{r}2275\\ 6500\\ \underline{32500}\\ 41275\end{array}$$

4) The final algorithm is:

$$\begin{array}{rl}325 & \\ \underline{\times 127} & \\ 2275 & \text{Multiply by 7}\\ 6500 & \text{Record zero to hold ones place, then multiply by the number of tens (2)}\\ \underline{32500} & \text{Record zeros to hold ones place and tens place, and then multiply by}\\ 41275 & \quad\text{the number of hundreds}\end{array}$$

c. Consider 302 x 416 or $\begin{array}{r}416\\ \underline{\times 302}\end{array}$

$$1)\ \begin{array}{r}416\\ \underline{\times\quad 2}\\ 832\end{array}\qquad\begin{array}{r}416\\ \underline{\times 300}\\ 124800\end{array}\qquad\begin{array}{r}832\\ \underline{124800}\\ 125632\end{array}$$

2) The final algorithm is:

$$\begin{array}{rl}416 & \\ \underline{\times 302} & \\ 832 & \text{Multiply by 2}\\ \underline{124800} & \text{Record zeros to hold ones place and tens place and then multiply by}\\ 125632 & \quad\text{the number of hundreds (3)}\end{array}$$

When the multiplier has no tens, we multiply immediately by the number of hundreds.

d. Consider 230 x 318 or $\begin{array}{r}318\\ \underline{\times 230}\end{array}$

$$1)\ \begin{array}{r}318\\ \underline{\times\ 30}\\ 9540\end{array}\qquad\begin{array}{r}318\\ \underline{\times 200}\\ 63600\end{array}\qquad\begin{array}{r}9540\\ \underline{63600}\\ 73140\end{array}$$

2) The final algorithm is:

$$\begin{array}{rl}318 & \\ \underline{\times 230} & \\ 9540 & \text{Since there are no ones, we multiply immediately by the number of tens}\\ \underline{63600} & \text{Then multiply by the number of hundreds (2)}\\ 73140 & \end{array}$$

## II. Practice

A. Replace the frames.

1. $3\times(6+4) = (3\times6) + (3\times\square)$
2. $2\times(1+\square) = (2\times1) + (2\times5)$
3. $15\times(\square+\triangle) = (15\times2) + (15\times3)$
4. $(\square\times4) + (\triangle\times4) = (2+3)\times4$
5. $(6+2) + (\square\times1) = 6\times(2+1)$

B. Rewrite each of the following using the distributive property. Find the products.

1. $(5+6)\times8$
2. $(2\times9) + (2\times3)$
3. $2\times(6+11)$

C. Replace the frames.

$$\begin{aligned} 26\times32 &= (\square+6)\times31 \\ &= (20\times\triangle) + (6\times\triangle) \\ &= \hexagon + \Box \\ &= 832 \end{aligned}$$

D. What is the meaning of each of the circled numerals?

1. $\begin{array}{r} 31 \\ \underline{\times29} \\ 279 \\ \underline{\textcircled{62}\phantom{0}} \\ 899 \end{array}$
2. $\begin{array}{r} 105 \\ \underline{\times\ 73} \\ 315 \\ \underline{\textcircled{735}\phantom{0}} \\ 7665 \end{array}$
3. $\begin{array}{r} 2014 \\ \underline{\times\ \ 39} \\ 18126 \\ \underline{\textcircled{6042}\phantom{0}} \\ 78546 \end{array}$

E. Multiply, using the vertical form. In each case, tell what two numbers are multiplied to obtain the first partial product; the second partial product.

1. $\begin{array}{r} 74 \\ \underline{\times63} \end{array}$
2. $\begin{array}{r} 89 \\ \underline{\times19} \end{array}$
3. $\begin{array}{r} 53 \\ \underline{\times11} \end{array}$
4. $\begin{array}{r} 236 \\ \underline{\times\ 45} \end{array}$
5. $\begin{array}{r} 406 \\ \underline{\times\ 38} \end{array}$

F. Rewrite your computations in C-1 and in C-2 using the expanded form.

G. Compute the products.

1. $\begin{array}{r} 605 \\ \underline{\times323} \end{array}$
2. $\begin{array}{r} 674 \\ \underline{\times405} \end{array}$
3. $\begin{array}{r} 3497 \\ \underline{\times\ 800} \end{array}$
4. $\begin{array}{r} 1754 \\ \underline{\times\ 176} \end{array}$

## III. Summary

A. What property of whole numbers does $6\times(5+1)=(6\times5)+(6\times1)$ illustrate?

B. What property of whole numbers do we use when we multiply by a factor greater than 10?

C. If the tens digit is zero in a three-place multiplier, how many partial products are there? (If only the tens digit is zero, there are two partial products. If the ones and tens digits are zero, there are no partial products.)

## Lesson 24

Topic: Division of Whole Numbers

Aim: To extend understanding of the short form of division by a number less than 10

Specific Objectives:

Review of long form of division by whole numbers less than 100
Understanding of and skill in using the short form of division by whole numbers less than 10

Motivation: When asked to divide 484 by 4, Mary showed her work as:

$$\begin{array}{r|r} \begin{array}{r} \underline{121} \\ 4)484 \end{array} & \\ \underline{400} & 100 \\ 84 & \\ \underline{80} & 20 \\ 4 & \\ \underline{4} & \underline{1} \\ 0 & 121 \end{array}$$

John, on the other hand, quickly wrote:

$$\begin{array}{r} \underline{121} \\ 4)484 \end{array}$$

How was John able to find the answer more quickly than Mary?

I. Procedure

A. Review long form of division by numbers less than 100.

Note: If pupils cannot do division by a two-place divisor, then they should be taught this using the following procedures:

1. Review multiplication by 10 and multiples of 10; by 100 and multiples of 100.

2. Consider:

$$15\overline{)360}$$

a. Which number is the dividend? the divisor?
b. How many 15's are there in 360? Have pupils estimate:

$10 \times 15 = 150$ (too small)
$20 \times 15 = 300$ (too small)
$30 \times 15 = 450$ (too large)

The answer (quotient) is greater than 20 and less than 30.

c. Computation:

$$\begin{array}{r|l} \multicolumn{1}{r}{24} & \\ \multicolumn{1}{r}{15\overline{)360}} & \\ \underline{300} & 20 \\ 60 & \\ \underline{60} & \underline{4} \\ 0 & 24 \end{array}$$

**Note** **to** **Teacher**: If some pupils have difficulty with estimation, the same result can be obtained by using any multiplication facts they know. For example:

$$\begin{array}{r|l} \multicolumn{1}{r}{24} & \\ \multicolumn{1}{r}{15\overline{)360}} & \\ \underline{150} & 10 \\ 210 & \\ \underline{150} & 10 \\ 60 & \\ \underline{60} & \underline{4} \\ 0 & 24 \end{array}$$

What is the meaning of the 2 in the quotient?
What is the meaning of the 4 in the quotient?
How may we check the answer?

3. Practice:

a. $7\overline{)357}$

b. $23\overline{)391}$

c. $54\overline{)1512}$

d. $38\overline{)954}$

e. $36\overline{)4356}$

B. Extend understanding of short form of division by numbers less than 10

1. Short form of division without regrouping

a. Refer to challenge: $4\overline{)484}$

Have pupils estimate the quotient: between 100 and 200.

b. Have pupils expand 484 and compute as follows:

$$\begin{array}{r} 100 + 20 + 1 \\ 4\overline{)400 + 80 + 4} \end{array} \qquad \text{or} \qquad \begin{array}{r} 121 \\ 4\overline{)484} \end{array}$$

c. How does the quotient compare with the estimate?

How would you verify your answer? (verify by multiplication: 4 x 121 = 484)

d. Have pupils practice:

Note: The expanded form of division may be discontinued when pupils are ready.

$2\overline{)264}$ $\qquad 3\overline{)693}$ $\qquad 5\overline{)550}$ $\qquad 8\overline{)808}$

2. Short form of division (involving regrouping)

a. Have pupils consider: $6\overline{)786}$

Have pupils estimate the quotient: more than 100.

b. Have pupils expand 786 and compute as follows:

$$6\overline{)700 + 80 + 6} \qquad \text{or} \qquad 6\overline{)786}$$

Since 700 is not divisible by 6, we regroup:

$$6\overline{)600 + 180 + 6} \qquad \text{or} \qquad 6 \begin{array}{|ccc} \hline 6 & 18 & \\ \cancel{7} & \cancel{8} & 6 \end{array}$$

$$\begin{array}{r} 100 + \ \ 30 + 1 \\ 6\overline{)600 + 180 + 6} \end{array} \qquad \text{or} \qquad \begin{array}{cccc} & 1 & 3 & 1 \\ \cline{2-4} 6 & \multicolumn{1}{|c}{6} & 18 & 1 \\ & \multicolumn{1}{|c}{\cancel{7}} & \cancel{8} & \cancel{6} \end{array}$$

c. How does the answer compare with the estimate?
How would you verify your answer?

d. Have pupils practice a variety of examples involving regrouping.

$4\overline{)568}$ $\qquad 7\overline{)840}$ $\qquad 9\overline{)1899}$ $\qquad 8\overline{)7280}$

$6\overline{)732}$ $\qquad 8\overline{)1064}$ $\qquad 5\overline{)650}$ $\qquad 9\overline{)34218}$

## II. Practice

A. In each of the following, show the regrouping of the dividend needed to use the short form of division:

1. $2\overline{)546}$ ( $2\overline{)400 + 140 + 6}$ )

2. $3\overline{)546}$ ( $3\overline{)300 + 240 + 6}$ )

3. $4\overline{)540}$

4. $5\overline{)540}$

B. Use examples and problems leading to computation similar to exercises in 2-d.

## III. Summary

A. In using the short form of division, it is often necessary to "regroup." What does this mean?

B. What is the advantage of knowing how to use the short form of division by whole numbers?

C. What mathematical vocabulary did we review today?

(dividend, divisor, quotient, regrouping, short form of division)

## Lessons 25 and 26

**Topic:** Division of Whole Numbers

**Aim:** To learn to divide by numbers greater than 99

**Specific Objectives:**

To learn to estimate the quotient
To learn to divide by a number greater than 99

**Motivation:** The school librarian reported that in a period of three months 563 seventh grade pupils had borrowed 3378 books.

What would you estimate is the average number of books borrowed by each pupil?

I. Procedure

A. Estimating the quotient

1. Elicit that to solve the motivation problem, the operation of division must be used.

2. Review terms: dividend, divisor, quotient, remainder.

3. Review multiplication of a number by 10 and multiples of 10; by 100 and multiples of 100.

4. Review the method of estimating the quotient.

a. Consider: $563\overline{)3378}$ (Motivation)

$1 \times 563 = 563$ (too small)
$10 \times 563 = 5630$ (too large)

The quotient is a number between 1 and 10.

Since 3378 is a little more than half of 5630 (10x563), the estimate is about 6.

b. Consider: $109\overline{)2639}$

$1 \times 109 = 109$ (too small)
$10 \times 109 = 1090$ (too small)
$100 \times 109 = 10900$ (too large)

The quotient is between 10 and 100.

Pupils try 20 x 109 = 2180 (too small)
30 x 109 = 3270 (too large)

The estimate then is more than 20, but less than 30.

c. Use similar procedure to have pupils estimate quotients for:

$918\overline{)35721}$ and $418\overline{)12540}$

5. Alternate method of estimating the quotient.

   a. Consider: $563\overline{)3378}$ (Motivation)

      Estimate: How many times are 5 hundreds contained in 33 hundreds? (6) The estimate is 6.

   b. Consider: $109\overline{)2616}$

      Estimate: How many times is 1 hundred contained in 26 hundreds? (26) The estimate is 26.

   c. Consider: $918\overline{)35721}$

      Estimate: How many times are 9 hundreds contained in 357 hundreds? (less than 40) The estimate is less than 40.

   d. Consider: $418\overline{)12540}$

      Estimate: How many times are 4 hundreds contained in 125 hundreds? (30) The estimate is 30.

B. Division by a three-place divisor

1. Divide: $563\overline{)3378}$

   a. The estimated quotient was 6. Enter 6 in ones' place in the quotient and proceed as in division by a two-place divisor.

$$\begin{array}{rl} 6 & \\ 563\overline{)3378} & \\ \underline{3378} & (6 \times 563) \\ 0 & \end{array}$$

      The quotient is 6. (The average number of books taken by each pupil was 6.)

   b. Have pupils verify the solution by multiplying.

2. Divide: $109\overline{)2639}$

a. The estimate was more than 20.

Enter 2 in tens place in the quotient.

$$\begin{array}{r l} 2 & \\ 109\overline{)2639} & \\ \underline{2180} & (20\times109) \\ 459 & \end{array}$$

b. Pupil compares 109 and 459.

$4 \times 109 = 436$ (too small)
$5 \times 109 = 545$ (too large)

The next partial quotient is between 4 and 5. Enter 4 in ones place.

$$\begin{array}{r l} 24 & \\ 109\overline{)2639} & \\ \underline{2180} & (20\times109) \\ 459 & \\ \underline{436} & (\ 4\times109) \\ 23 & \end{array}$$

The solution is 24 and remainder 23.

Compare with estimate.

Note: The writing of 0's may be omitted when pupils show understanding of the division process.

c. Check by computing the product of the quotient and the divisor and adding the remainder to this product.

3. Divide: $918\overline{)35721}$

a. The estimate was less than 40.

Enter 3 in tens place in the quotient.

$$\begin{array}{r} 3\phantom{1} \\ 918\overline{)35721} \\ \underline{2754}\phantom{1} \\ 8181 \end{array}$$

b. Pupil compares 918 and 8181.

$$\begin{array}{r} 38 \\ 918\overline{)35721} \\ \underline{2754}\phantom{1} \\ 8181 \\ \underline{7344} \\ 837 \end{array}$$

$9 \times 918 = 8262$ (too large)
$8 \times 918 = 7344$ (too small)

The next partial quotient is between 8 and 9. Enter 8 in ones place.

The solution is 38 and remainder 837.

Compare with estimate.

c. Check as in 2-c.

4. Divide: $418\overline{)12540}$

$$\begin{array}{r} 3\phantom{0} \\ 418\overline{)12540} \\ \underline{1254}\phantom{0} \end{array}$$

a. The estimate was 30.

Enter 3 in tens place in the quotient.

$$\begin{array}{r} 30 \\ 418\overline{)12540} \\ \underline{1254}\phantom{0} \end{array}$$

b. Elicit that

although there is a remainder of zero, the digit 3 is in tens place. Therefore, the quotient is 30.

Compare with estimate.

c. Check.

II. Practice

A. Estimate the quotient:

1. $59\overline{)4704}$
2. $65\overline{)1857}$
3. $26\overline{)64300}$
4. $121\overline{)4629}$
5. $320\overline{)4080}$
6. $138\overline{)42785}$
7. $511\overline{)25720}$
8. $207\overline{)3726}$

B. What is the value of each circled numeral?

$$\begin{array}{r} 411 \\ 46\overline{)18948} \\ \underline{\enclose{circle}{184}}\phantom{00} \\ \enclose{circle}{54}\phantom{0} \\ \underline{\enclose{circle}{46}}\phantom{0} \\ 88 \\ \underline{46} \\ 42 \end{array}$$

C. Estimate, divide and check:

1. $144\overline{)3312}$
2. $322\overline{)3864}$
3. $240\overline{)15120}$
4. $406\overline{)32886}$
5. $206\overline{)5866}$
6. $299\overline{)10001}$
7. $321\overline{)19260}$
8. $705\overline{)217845}$
9. $301\overline{)7250}$
10. $119\overline{)36547}$

III. Summary

A. How do you estimate the quotient in a division example?

B. How can estimating help you decide whether a quotient is 6 or 60?

C. If a divisor is 179, what is the greatest remainder? the least remainder?

D. How can we check a division example?

## Lessons 27 and 28

Topic: Multiplication and Division of Whole Numbers

Aim: To learn to solve problems involving multiplication and division of whole numbers

Specific Objectives:

To reinforce the meaning of multiplication
To reinforce the meaning of division
To analyze and solve problems involving multiplication and division

Motivation: In solving a word problem, how can you tell when multiplication is to be used? when division is to be used?

I. Procedure

A. Review the meaning of multiplication

1. Have pupils consider the following problem:

   Allan's stamp album has room for 15 stamps on each page. If there are 40 pages in the album, how many stamps can it contain in all?

2. Elicit that in this problem we have 40 equivalent disjoint sets. There are 15 elements in each set.

3. Have pupils recall that to find the total of the numbers of two or more equivalent disjoint sets, we may use multiplication.

$$40 \times 15 = \square \quad \text{or} \quad \begin{array}{r} 40 \\ \underline{\times 15} \\ \square \end{array}$$

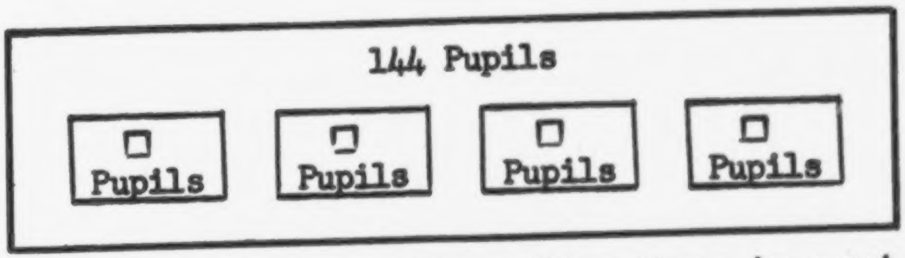
   There is room for 600 stamps in the album.

B. The meaning of division

1. Consider the problem:

   There are 144 pupils in four seventh-grade classes. The number on register in each of the classes is the same. How many pupils are in each class?

   a. Diagram the situation.



b. Elicit that in this problem there is a set (whose number is 144) and 4 equivalent subsets of this set. What is the size of each of the equivalent subsets?

   1) What arithmetic operation is used to find the size of each of the equivalent subsets? (division)

   2) How do we express this as a mathematical sentence?

      $144 \div 4 = \square$ or $\begin{array}{r}\square\\4\overline{)144}\end{array}$

      There are 36 pupils on register in each class.

2. Consider the problem:

   There are 144 pupils in all of the seventh-grade classes in a school. If there are 36 pupils on register in each class, how many seventh-grade classes are there?

   a. Elicit that in this problem there is a set (whose number is 144), and 36 elements in each of a certain number of equivalent subsets of this set. How many equivalent subsets are there?

   b. What arithmetic operation is used to find the number of equivalent subsets of a set? (division)

   c. How do we express this as a mathematical sentence?

      $144 \div 36 = \square$ or $\begin{array}{r}\square\\36\overline{)144}\end{array}$

      There are 4 seventh-grade classes.

3. Have pupils see that division is used:

   a. to find the size (number) of each subset when a set is separated into a given number of equivalent subsets. (motivation)

   b. to find the number of equivalent subsets of a given set when we know the number of elements in each subset. (motivation)

4. Have pupils state the operation(s) to be used in solving each of the following problems:

   a. How many are there in 6 sets of 7?
   b. From a set of 50 remove a set of 22. How many are left?

c. Arrange a set of 54 into 9 equivalent subsets.
How many are there in each subset?

d. There are 5 sets of 7 and one set of 6.
How many are there in all?

e. How many more are there in 6 sets of 8 than in one set of 9?

f. Arrange a set of 72 into 6 equivalent sets.
How many are there in 4 of these subsets?

g. Arrange a set of 63 into 7 equivalent sets.
How many are there in a set of 10 together with 4 of the seven subsets above?

h. △ cookies are to be packed into □ boxes.
Each box is to contain the same number of cookies.
How many cookies will there be in each box?

  1) What does the symbol △ represent? (the total number of cookies, i.e., the size of the given set)

  2) What does the symbol □ represent? (the total number of boxes, i.e., the number of equivalent groups or subsets in the given set)

  3) What are we to find? (the number of cookies to go into each box, i.e., the size of each of the equivalent groups or subsets)

  4) What operation should we use? (division)

i. Have pupils suggest numbers and solve the problem.

C. Analyzing problems

1. Recall steps in analyzing a problem.

   a. What is to be found?
   b. What information is given?
   c. What operation(s) shall we use?

2. Consider the problem:

Jim Carmen wants to buy a guitar. The guitar which is priced at $63 is made of fine wood and is a bright brown. Mr. Robins, who owns the store, is a neighbor of Jim's. Mr. Robins usually requires a down payment of $10. He is willing to sell the guitar to Jim without the down payment because he likes Jim. He tells Jim that he may pay $7 a month. How many months will it take Jim to pay for the guitar?

a. What is to be found? (the number of months it will take Jim to pay for the guitar)

b. What facts are given that are needed to arrive at a solution? (the cost of the guitar, the amount of each payment)

c. What operation(s) will we use to relate the given facts to what we wish to find? (division, since we are given the size or number of a set, 63, and we are to find the number of equivalent subsets each of which has 7 elements)

$$63 \div 7 = \square \quad \text{or} \quad \begin{array}{r} \square \\ 7\overline{)63} \end{array}$$

Jim will make payments for 9 months.

d. What facts are irrelevant? (the $10 down payment, the description of Jim as a neighbor, the description of the guitar)

3. Consider the problem:

A school has 12 seventh-grade homeroom classes with 34 pupils in each class. If each pupil takes typing, how many typing classes must be formed if there are to be 24 pupils in each class?

a. What is to be found? (the number of typing classes to be formed)

b. What facts are given? (the number of homeroom classes; the number of pupils in each homeroom class; the number of pupils to be included in each typing class)

c. What operations will we use to relate the given facts to what we wish to find? (multiplication - why? ; division - why?)

$$12 \times 34 = 408$$

$$\begin{array}{r} 17 \\ 24\overline{)408} \\ \underline{24\phantom{8}} \\ 168 \\ \underline{168} \\ 0 \end{array}$$

There will be 17 typing classes formed.

II. Practice

A. If shirts cost a storekeeper $36 per dozen, how much does he pay for a shipment of 134 shirts?

B. When Martin's father bought his new car, he made a down payment of $842. He paid the balance in 18 monthly installments of $94 each. What was the total cost of the car?

C. The odometer on Mr. Johnson's car read 19,217 miles when he left home at 9 a.m. If he can average 41 miles an hour, what should be the odometer reading at 12 noon?

D. The population of a certain city increased from 14,462 to 15,977 over a 15-year period. What was the average increase per year?

E. Consult suitable textbooks for additional problems.

III. Summary

A. Elicit the steps in analyzing a problem.

B. When is the operation of multiplication used in solving a problem?

C. When is the operation of division used in solving a problem?

# CHAPTER III

The material in this chapter is intended to help the teacher develop the pupil's understanding of the relationship between the physical world and geometry. This section contains suggested procedures for developing concepts of:

- point
- space
- line
- line segment
- ray
- angle
- simple closed curve
- parallel lines
- space figure

A careful distinction is made in these materials between abstract geometric concepts and the pictures that we draw to represent these concepts. The kind of distinction that was made between numbers and numerals is also made in the geometric context between a geometric idea such as a point or a line, and the picture of a point or line drawn on paper or on the chalkboard.

The language of sets with which the pupil is now familiar is used to describe many of the geometric concepts. For example, space is defined as the set of all points; an angle is defined as the union of two rays with a common endpoint, and so on.

The use of the discovery approach is suggested to help the pupil become aware of some of the basic principles out of which the study of geometry develops. By means of this approach, pupils are guided to a realization that two points determine a unique line; two distinct intersecting planes have exactly one common line; a line not in a plane and intersecting a plane has just one point in common with the plans, and so on.

The basic geometric concepts and relationships presented in this chapter will provide useful background for the pupil as he engages in a systematic study of geometry in future years.

# CHAPTER III

## NON-METRIC GEOMETRY

Lessons 29-40

### Lesson 29

Topic: Space and Points

Aim: To reinforce and extend the meaning of point and space

Specific Objectives:

A point is a mathematical idea
A point has no size
A point is a location or position in space
Naming a point
Geometric figures

Motivation: More than 2000 years ago, a mathematician named Archimedes tried to compute the number of grains of sand it would take to fill all of space.

How large a number do you think he arrived at as the result of his calculations?

I. Procedure

A. A point is a mathematical idea

1. Tell pupils that Archimedes' calculations resulted in a number named by writing 1 with 63 zeros after it.

2. Elicit that the number computed by Archimedes (as well as any other number) is an idea that exists in our imagination. We represent this number by a numeral.

3. Discuss with pupils that we may imagine all of space filled, not with grains of sand, but with points. This is the kind of space we study in geometry: a space filled with points. However, just as we never see a number, but only its representation by a numeral, so also we never see a point. A point is a mathematical idea. It exists only in our imagination.

4. A dot and a point of a pencil each suggest a representation of a point. What other representation of a point can you think of? (a spot on the wall, a star, a particle of dust in the air, the point where the ceiling and two adjoining walls meet)

### B. A point has no size

1. Imagine a point in space. Represent this point by drawing a dot on your paper.

2. Represent a second point in space by marking a larger dot on your paper. Is the second point larger than the first?

3. Tell pupils that in our mathematical world we say a point has no size.

4. We may imagine a geometric point as being smaller than anything in the world. How many points on the desk are covered by a book placed on the desk?

### C. A point is a location or position in space

1. How is the location of New York City indicated on a map? (a point on the map)

2. When a ship radios its position, how does the Coast Guard indicate the ship's position on the map? (a point)

3. What star in the sky have mariners always consulted in steering their course? (North Star) Why? (definite position in the sky)

4. Elicit that a point is a location or position in space.

5. Have pupils think of various points in the classroom. Have them describe the location of these points so that their classmates can find the points.

### D. Naming a point

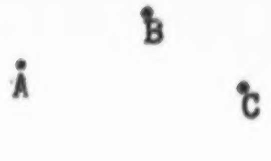
1. Mark three models of points (dots) on your paper. How can we differentiate among them?

2. Elicit that just as we used names to differentiate among pupils in the classroom, so also we can differentiate among points if we name them.

3. Tell pupils that a point is named by a capital letter placed near the dot representing the point.

4. How many points are pictured?

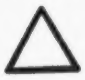
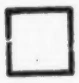
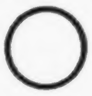
   What are the names of the points?



Note: Explain to pupils that hereafter we shall say, "Draw a point" but we shall always mean, "Draw the model of a point."

E. Geometric figures

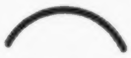
1. Consider these different sets of points:

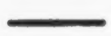




Name the geometric figure formed by each.

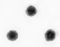
2. Tell pupils that any set of points is a geometric figure.

3. Which of the following represent geometric figures?

a. 

c. 

b. {a,b,c}

d. 

II. Practice

A. Tell which words in the list describe physical things and which describe ideas.

1. New York City
2. Sunlight
3. Bravery
4. The numeral 8
5. The number 8
6. Helpfulness
7. Dot
8. Point

B. Draw a point on your paper. Draw a point to the right of it; to the left of it; above it; below it.

1. In what way do these points differ? (position)

2. Of what set are these points a subset? (all points on the paper; points in space)

3. Name the points you have drawn.

C. Think of a point in the classroom. Describe its location so that your classmates can find the point.

D. How many representations of a point do you think you can place on your paper to cover it completely? on the chalkboard? on the wall?

E. How many points are there in the room? the building? in all of space? (an infinite number in each case)

F. Look at this "empty" box. In terms of mathematics, with what is it "filled"? Let's move the box to another place on the desk. With what is it now filled?

Explain why these points are not the same as the points in the box when it was in its previous place on the desk. (Points are positions. These are different positions.)

G. When NASA decides that two rockets will rendezvous, what does this mean? (They will meet at the same point in space.)

III. Summary

A. In geometric space, what is a point? (an idea, a position)

B. How may we represent a point? (by a dot)

C. Distinguish between a point and a dot. (A point is a mathematical idea which exists only in our imagination; a dot is a model or representation of a point.)

D. How large is a point? (A point has no size.)

E. How many points are there in space? (an infinite number)

F. How do you name the model of a point?

G. What is meant by a geometric figure?

## Lesson 30

Topic: Line Segments and Lines

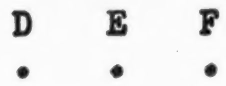
Aim: To reinforce and extend the meaning of line segment

Specific Objectives:

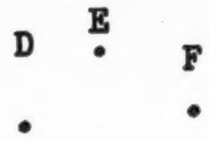
Meaning of line segment
Naming a line segment; symbol for line segment
A line segment has one dimension - length
Two points determine a line segment

Challenge: In which of the following sets of three points is point E between D and F?

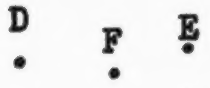
a.



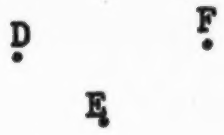
b.



c.



d.



I. Procedure

A. Concept of a line segment

1. Have pupils see that only in a is point E between D and F. (challenge)

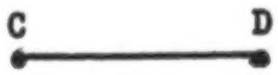
   Note: Point E is said to lie between points D and F if and only if all three points are distinct points on a line and DE + EF = DF.

2. Have pupils draw two points and label them C and D. Have them draw another point between C and D and label it E. Have them draw as many points as they can between points C and D.

   C.....E.....D

   How many points are there between C and D? (an infinite number)

3. Tell pupils that the two points C and D and all the points between them make up a set of points called a line segment. The two points C and D are called the endpoints of the line segment.



4. Have pupils give examples of models of line segments that they see about them: edge of the desk, edge of a ruler, and so on.

B. Naming a line segment; symbol for line segment

1. Have pupils suggest a way to name the picture of a line segment which each has drawn, as, for example,

This line segment is called "line segment AB" or "line segment BA."

2. The symbol we use for the words "line segment" is a bar over the capital letters, i.e., $\overline{AB}$. This symbol is read: "line segment AB."

What is another symbol for the same line segment? ($\overline{BA}$)

How is this symbol read?

C. A line segment is one-dimensional

1. How is your picture of a line segment different from those of your classmates? (longer, shorter, the same length)

2. Elicit that a line segment has only one dimension - length.

D. Two points determine a line segment

1. Have pupils draw several pairs of points and have them join each pair with a straight edge. In each case, what geometric figure is drawn? (a line segment)

2. For any two points, how many line segments are there which consist of the two points and all the points between them? (only one)

3. Tell pupils that since for any two points there is only one line segment, we say that two points determine a line segment.

## II. Practice

A. Write in symbols:

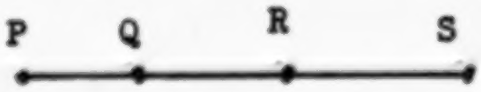
1. line segment BC
2. line segment FG
3. line segment PQ

B. What is the meaning of the following symbols?

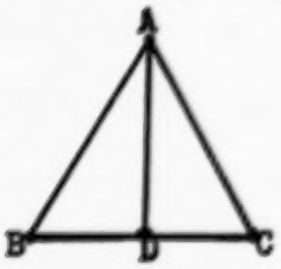
1. $\overline{MN}$
2. $\overline{CD}$
3. $\overline{XY}$

C. Make a dot to represent point P on your paper. Draw pictures of four line segments which have P as one endpoint.

D. Name six line segments that are represented below.



E. Name as many line segments as you see represented in the drawing.



1. Which point is contained in both $\overline{BD}$ and $\overline{DC}$?
2. Which point is contained in both $\overline{AB}$ and $\overline{AC}$?

## III. Summary

A. What is the meaning of a line segment?
B. How many line segments are determined by any two points?
C. How is a line segment named? What symbol is used for line segment?
D. How many dimensions does a line segment have? What is this dimension?
E. What new vocabulary have you learned today? (line segment, endpoint)

## Lesson 31

Topic: Line Segments and Lines

Aim: To reinforce and extend the meaning of line

Specific Objectives:

Concept of a line
Naming a line; symbol for line
The number of lines through a point
Two points determine a line
Two different lines do not intersect in more than one point

Challenge: We have been using the word "segment" which means a piece or a part. Consider the following picture of a line segment:

A.______.B

Of what do you think $\overline{AB}$ is a piece or a part?

I. Procedure

A. Concept of a line

1. With reference to the challenge, elicit that $\overline{AB}$ is a piece or a part of a line. How shall we draw the line?

2. Have pupil extend the line segment in one direction and then in the opposite direction. How far can this picture of a line be extended in each direction on your paper? How far can a line (the idea) be extended in each direction? (endlessly)

   What is the difference between a line and a line segment? (a line has no endpoints; a line segment has two endpoints)

3. Have pupils try to give examples of representations of lines in the physical world. (In the physical world there are representations of line segments only.)

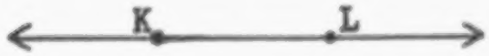
B. Naming a line; symbol for a line

1. Draw a picture of line segment KL with endpoints K and L.

   K._______.L     Extend it in each direction     ____K._____.L____

   Of what does it now appear to be a model? (a line)

   How can we indicate that the line is unending? Tell pupils that arrowheads are drawn to show that the line is unending.

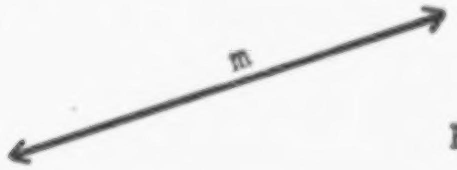


2. We name the line represented above, line KL.

   We shall write $\overleftrightarrow{KL}$ to mean "line KL."

   Another name for this line is $\overleftrightarrow{LK}$. This is read "line LK."

3. A line may also be named by a lower case letter.



Read "line m"

C. The number of lines through a point

1. Make a dot on your paper to represent point D.

   a. Draw a picture of a line r which passes through point D.

   b. Draw a picture of a line s which passes through point D.

   c. Draw a picture of a line t which contains point D.

   d. How many lines pass through point D?

2. Have pupils conclude that for any point there is an infinite number of lines which contain this point.

D. Two points determine a line

1. Mark two dots on your paper to represent points E and F.

   a. Draw a picture of a line which contains points E and F.

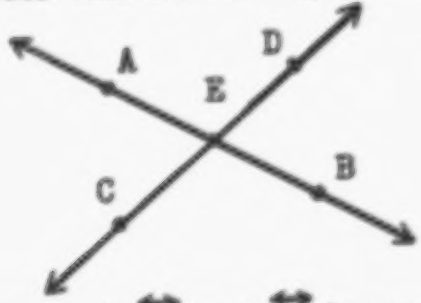
   b. Try to draw another picture of a line which contains points E and F. Can this be done?

2. Have pupils conclude that for any two points there is exactly one line which passes through both of the given points.

   We sometimes express this thought by saying that two points determine a line.

E. Two different lines do not intersect in more than one point

1. Consider the following drawing:



Elicit that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have point E in common. When two lines have a point in common we say the lines intersect.

2. Have pupils draw several pictures of pairs of lines. What is the greatest number of points in which any pair of different lines intersect?

3. Have pupils conclude that two different lines do not intersect in more than one point.

II. Practice

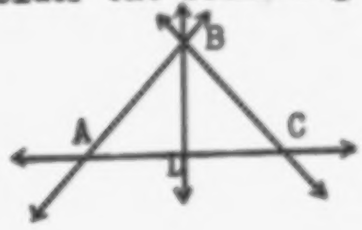
A. Write in symbols:

1. line MN
2. line XY

B. What is the meaning of the following symbols?

1. $\overleftrightarrow{AB}$
2. $\overleftrightarrow{PQ}$

C. Consider the following drawing:



1. Name four lines in the drawing using two points to name each line.

2. Are $\overleftrightarrow{AD}$, $\overleftrightarrow{DC}$ and $\overleftrightarrow{AC}$ different lines? Explain.

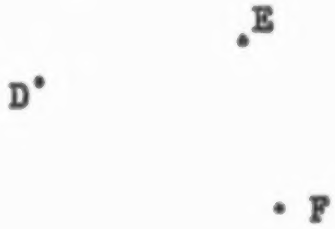
3. What point is contained in both $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$? In what point do these lines intersect?

4. How many points on $\overleftrightarrow{BD}$ are named? How many points are there on $\overleftrightarrow{BD}$? (a limitless number)

D. Mark on your paper three dots which represent points D, E, and F.

Draw as many pictures of lines as you can which contain two of the points.

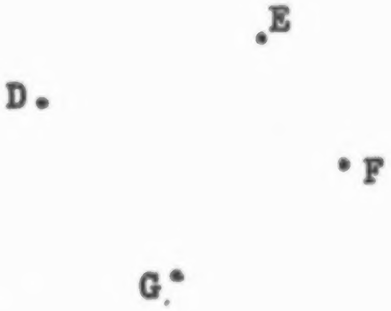
How many pictures of lines did you draw?



E. Mark on your paper four dots which represent points D, E, F, and G.

Draw as many pictures of lines as possible which contain two of the points.

How many pictures of lines did you draw?



III. Summary

A. How is a line different from a line segment?

B. What are two ways of naming a line?

C. How many pictures of lines can you draw which contain the same two points R and S?

D. How many different lines contain a point A?

E. What is the greatest number of points in which any pair of different lines intersect?

F. What new vocabulary and symbols have you learned today?

(intersect, symbol for line)

## Lessons 32 and 33

Topic: Rays and Angles

Aim: To reinforce and extend the meaning of ray; of angle

Specific Objectives:

- Meaning of a ray
- Naming a ray; symbol for ray
- Meaning of angle
- Naming an angle; symbol for angle

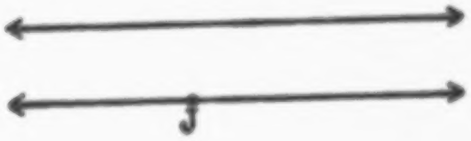
Motivation: We often see "rays" of light - from a flashlight, from an automobile headlight, from the sun. In each case, where does the ray start? How far does it extend?

What is a ray?

I. Procedure

A. Meaning of a ray

1. What geometric figure does this drawing represent? What is a line?

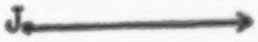


2. Consider point J on the line.

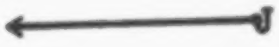
   a. Into how many sets of points has point J divided the set of points of which the line consists? (3)

   b. Describe each set. (The set of points on one side of J; the set of points on the other side of J; the set with the single element, point J)

3. Tell pupils that the set of points composed of J and all the points of the line on one side of J is called a ray. (motivation)

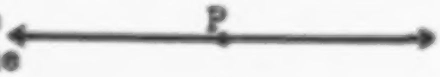


The set of points composed of J, and all the points on the line which are on the other side of J is also a ray.



J is called the endpoint of either ray.

a. How many rays are pictured in the figure on the right? (two - P and the part of the line on one side; P and the part of the line on the other side)

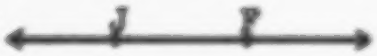


b. Why may we think of the "ray" from a flashlight as a model of a mathematical ray? (The flashlight may be considered the endpoint and the beam of light, the (half) line.)

4. Elicit that we may say a ray is the union of a given point on a line and all other points on the line on one side of this given point.

B. Naming a ray; symbol for ray

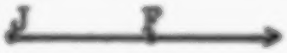
1. Consider the drawing at the right:



a. What is the name of this line?

b. How many rays are shown which contain point J as an endpoint?

2. Consider the ray consisting of endpoint J and all the points on the line to the right of J.



a. Tell pupils that we use the endpoint of a ray as well as another point on the ray to name the ray. (the endpoint is named first)

b. Thus, the ray pictured above may be named ray JF.

c. A simple way of writing ray JF is $\overrightarrow{JF}$.

3. Elicit that to name the other ray with endpoint J, we must name a point to the left of J, for example, point K.

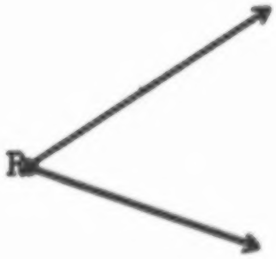
a. This ray is called ray JK.

b. The symbol for this ray is $\overrightarrow{JK}$.
Note that in the symbol the arrow ( → ) always points to the right.

C. Meaning of angle

1. Have pupils recall the meaning of union of two sets.
The union of two sets is a set which consists of only those elements which belong to either or both sets.

2. Have pupils mark a point on paper. Have them label the point R.



   a. Draw a ray with R as the endpoint.

   b. Draw another ray with R as the endpoint.

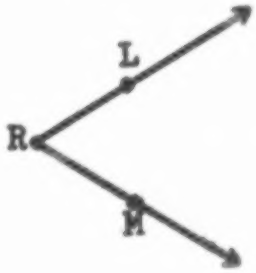
   c. What geometric figure is represented by your drawing? (an angle)

   d. How many rays are there in the drawing? What point do they have in common?

3. What elements are in the union of the two rays represented in the diagram in 2? (the endpoint R, all the other points in each ray)

4. Elicit that an angle is a set of points. It is the union of two rays, not on the same line, which have a common endpoint.

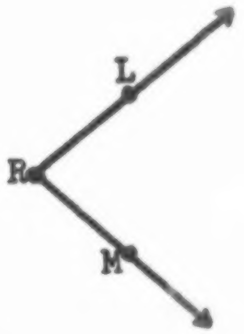
5. Have pupils recall that each ray is called a side of the angle. To name the sides, we name any point on each of the two rays.



   Name the two sides of the angle pictured above.

6. Have pupils recall that the common endpoint of the rays is called the vertex of the angle. What is the vertex of the angle in 5?

### D. Naming an angle; symbol for angle



1. Have pupils recall the several ways of naming an angle.

   a. The symbol for angle is ∠.

   b. The angle pictured above may be named angle LRM or ∠LRM. It may also be named ∠MRL.

   c. Where is the letter naming the vertex placed in naming an angle?

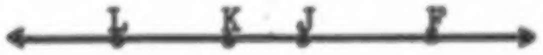
   d. An angle may also be named simply by its vertex letter, as, for example, angle R or ∠R.

2. Have pupils draw and name several angles.

II. Practice

A. Draw a picture of a line; of a line segment; of a ray.

B. Refer to the diagram at the right to answer the questions.



1. How many rays can you find represented in the diagram?
   Name all the rays you can identify.

2. Are $\overrightarrow{LK}$ and $\overrightarrow{LJ}$ names for the same rays?
   Explain.

3. Which point is contained in both $\overrightarrow{KJ}$ and $\overrightarrow{KL}$?

4. What is the intersection of $\overrightarrow{KJ}$ and $\overrightarrow{KL}$?

C. Draw the union of two rays not on the same line with a common endpoint.

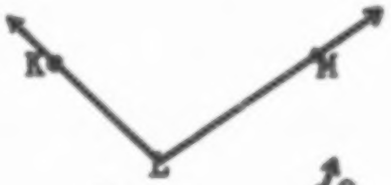
1. What is the geometric figure you have drawn called?

2. Label the rays. What is the common endpoint?

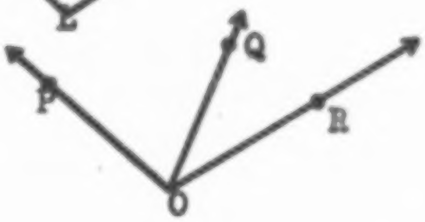
3. What is the name of the angle?

4. Name the vertex. Name the sides.

D. What is the name of this angle?
( $\angle KLM$, $\angle MLK$, $\angle L$ )



E. Consider the diagram at the right.



1. What is the common vertex of these angles? (O)

2. How many angles are shown in this diagram? (3)

3. Name the angles. ( $\angle POQ$, $\angle QOR$, $\angle POR$ )

4. Why do we not call any of these angles $\angle O$?

F. The hands of a clock suggest an angle.

1. Where is the vertex?

2. Why are the hands not exactly the model of a ray?

G. Describe five objects in your classroom or in your home which suggest angles.

H. If three rays, not on the same line, have a common endpoint, how many angles are formed?

I. If four rays, not on the same line, have a common endpoint, how many angles are formed?

III. Summary

A. How are a ray and a line related? (A ray is part of a line.)

B. What is the difference in the naming of a line and of a ray? (A line is named by any two points on it; a ray is named by its endpoint and any other point on it.)

C. What is the difference in the naming of a line segment and of a ray? (A line segment is named by its two endpoints; a ray is named by its endpoint and any other point on it.)

D. Consider the line, the line segment, and the ray.
Of which can you get a measure of length? (line segment)
Explain. (Only the line segment is not unending.)

E. How can we describe an angle in terms of the union of two rays?

F. When an angle is formed by the union of two rays, what is their common endpoint called?
What are the rays called?
Do the rays or sides of an angle ever end? Explain.

G. In naming an angle using three letters, what is the position of the letter which names the vertex?

H. What is another way of naming an angle?

I. What mathematical vocabulary have you used today?

(ray, angle, side, vertex)

## Lessons 34 and 35

Topic: Planes and Lines

Aim: To reinforce and extend understanding of the meaning of a plane

Specific Objectives:

To understand:
- A flat surface is a model of a plane
- The set of points in a plane extends without end
- If two points of a line are in a plane, the whole line is in the plane
- The intersection of a plane and a line which is not in the plane is a point
- If two planes intersect, their intersection is a line
- Three points determine a plane

Challenge: In the woodworking shop, an instrument called a plane is frequently used.

For what is it used?

I. Procedure

A. A flat surface is a model of a plane

1. Elicit that the instrument we call a plane is used in the woodworking shop to create a flat surface.

2. Tell pupils that in mathematics, just as we consider a dot to be a model of a point and a straight edge to be a model of a line, so we consider the set of points on a flat surface to be the model of a plane.

3. Have pupils suggest illustrations of models of planes: the desk top, the floor of the classroom, a piece of paper, and so on.

B. The set of points in a plane extends without end

1. Consider a page in your book. Of what is the page a model? (a plane)

2. Elicit that before the page was cut to this size, it was a portion of a larger sheet of paper. We can think of a still larger sheet of paper of which the page is a portion.

3. Imagine the sheet of paper growing longer and longer, and wider and wider, stretching without end.

   Elicit that the set of points in a plane extends without end.

4. Discuss with pupils the fact that the illustrations they have suggested for models of planes are actually representations of parts of planes.

C. If two points of a line are in a plane, the whole line is in the plane

1. Have pupils consider the part of a plane represented by a sheet of notebook paper.

   a. Have them mark two points on the paper and label them R and S.

   b. How many (straight) lines can you imagine containing the two points?

   c. Draw a picture of a line going through the dots representing points R and S.

      Will any points of the line containing R and S not be in the plane represented by the paper?

   d. Will all points of $\overleftrightarrow{RS}$ be in the plane containing R and S?

   e. Have pupils conclude that if two points of a line are in a plane, the whole line is in the plane.

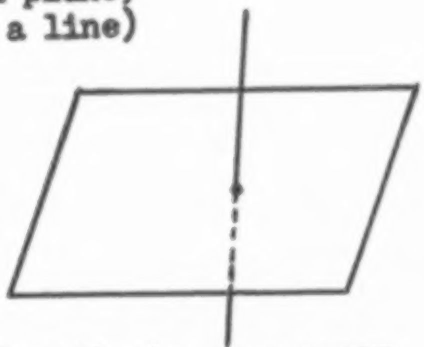
D. The intersection of a plane and a line which is not in the plane is a point

1. Have pupils consider a sheet of paper and a pencil.
   What does the paper represent? (a part of a plane)
   What does the pencil represent? (a part of a line)

2. Put the pencil through the paper.

   a. What would you call the intersection of the plane and the line? (a point)

   b. How many points do the plane and line have in common? (one and only one)



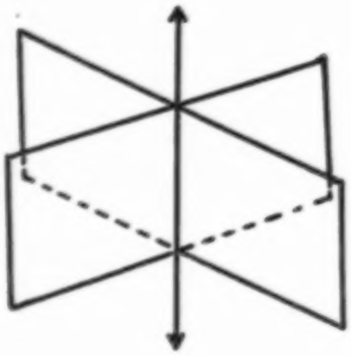
Pencil piercing paper

3. Elicit other illustrations of a line intersecting a plane. (a nail hammered into a wall)

4. Have pupils conclude that the intersection of a plane and a line which is not in the plane is a point. The point is in both the line and the plane.

E. If two planes intersect, the intersection is a line

1. Take two sheets of paper and cut a slit in one sheet.



Pass the other sheet of paper through the slit..

a. What geometric figure does each paper represent? (a part of a plane) What does the slit represent? (a line segment)

b. If the two sheets of paper represent planes, what is their intersection? (a line)

2. Consider the floor and a wall as representing parts of planes.

a. What would you call their intersection? (a line segment)

b. If the floor and the wall were planes, what would the intersection be?

3. After several such illustrations, elicit that if two planes intersect, their intersection is a line. The line is in both planes.

F. Three points determine a plane

1. Why must a chair or a table have at least three legs?

a. Have pupils consider what would happen were a chair to have only two legs.

Elicit that a chair with only two legs could not remain in a standing position.

b. Guide pupils to understand the mathematical reason for this as follows:

1) Consider the points of two pencils as points in space.
Use a cardboard to represent a part of a plane.
Holding the pencils still, touch the cardboard to the pencils.
Tilt the cardboard about the two pencil points, so that it assumes various positions.

2) What do the different positions of the cardboard represent? (different planes)

3) How many planes in space contain two given points? (a limitless number)

c. Place a cardboard on three pencil points not on the same line.

If the pencils are held in place, can the cardboard be tilted now? (No - there is only one position of the cardboard on the three pencil points.)

2. Have pupils conclude that one and only one plane contains three points which are not on the same line.

II. Practice

A. How can we decide whether a particular surface represents part of a plane?

1. Let the chalkboard surface represent a part of a plane.
Mark two points on the chalkboard and join them with a straight edge.

a. What geometric figure have you pictured? (a line segment)

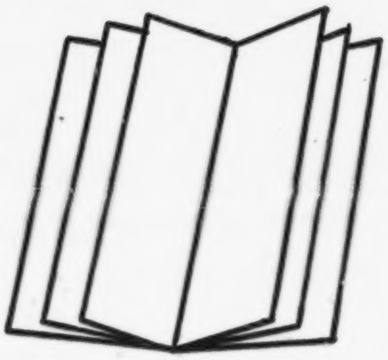
b. Does the line segment lie entirely in the plane of the chalkboard?

2. Mark two other points on the chalkboard.
Repeat the procedure outlined above and note again that the line segment lies entirely in the plane of the chalkboard.

3. If a line segment joining any two points of a surface lies entirely within the surface, what can you conclude about the surface? (it represents part of a plane)

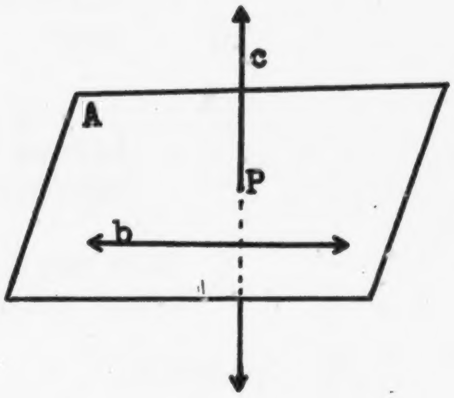
B. How would you prove that a drinking glass is not a flat surface?

C. Consider several pages of an open book as shown at the right.



1. What do these pages represent? (parts of planes)
2. What would you call the intersection?
3. Think of a line in space. How many planes can be thought of which contain the line?

D. Refer to the diagram below in which a part of a plane is represented. The plane has been named by a capital letter placed in the corner of its representation.



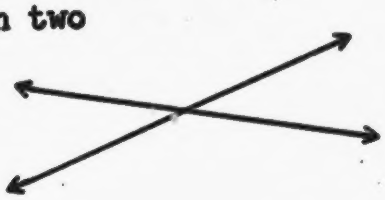
1. Where does line b lie? (on plane A)
2. How many planes can you think of which contain line b?
3. Does line c lie on plane A?
4. What is the geometric figure formed by the intersection of line c and plane A? (a point)
5. Name the intersection of line c and plane A. (point P)

E. Consider a shoe box

1. What geometric figures do the sides of the box represent? (parts of planes)
2. If the sides of the box were planes, what would the intersection of two sides be?
3. Are there any planes suggested by the sides of the box which do not intersect?

F. (Optional) How many planes will contain a given point and a given line?

G. (Optional) How many planes will contain two intersecting lines?



III. Summary

A. Why is a ball not an illustration of a portion of a plane?

B. Why is a sheet of paper called a model of part of a plane?

C. If two points of a line are in a plane, what can you conclude about all the other points of the line ?

D. What mathematical idea does a tie tack passed through a tie illustrate?

E. Why does folding a paper produce a straight edge?

F. Why will a table on three legs not rock?
Why may one with four legs rock?

G. What new vocabulary have you learned today? (plane)

## Lesson 36

**Topic: Closed Curves**

**Aim: To reinforce and extend understanding of a simple closed curve**

Specific Objectives:

Concept of closed curve
Meaning of simple closed curve
A simple closed curve separates the plane into three sets of points

Motivation: Sometimes when you are listening to a speaker you occupy your hands by making drawings on a piece of paper. We say you are doodling. Make such a drawing.

What familiar objects do your doodles resemble? (faces, flowers, squares, rectangles, circles, lines, numerals) Why do mathematicians consider all your doodles to be geometric figures?

I. Procedure

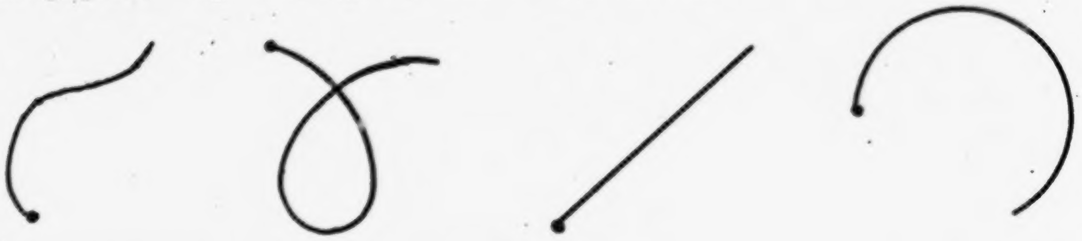
A. Concept of closed curve

1. Elicit that a drawing is a representation of a set of points. Since any set of points is considered to be a geometric figure, all doodles are considered to be geometric figures.

2. Draw a point. Starting at this point draw any figure in the plane of the paper without lifting your pencil from the paper.

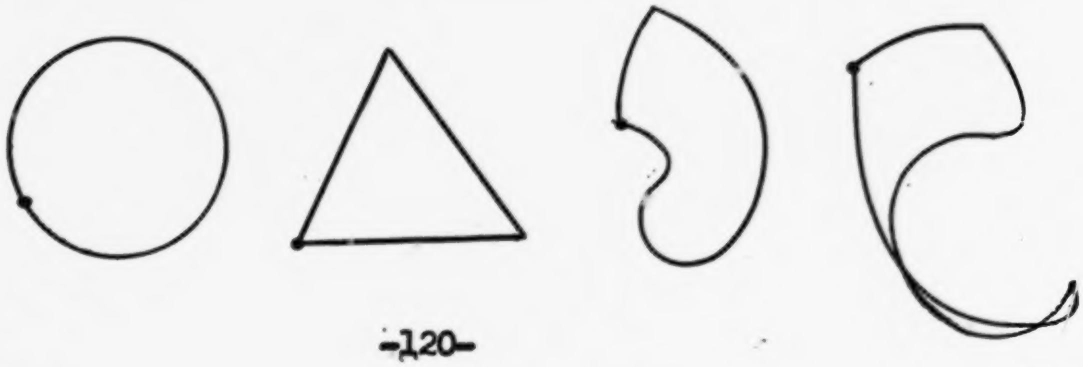
Some pupils may draw pictures like these:

a.



Others may draw figures like these:

b.



Tell pupils that if a picture of a set of points can be drawn without lifting the pencil from the paper, the figure is called a *curve*.

Thus, mathematicians use the word *curve* for *any* kind of line drawn in this way, whether straight or curved.

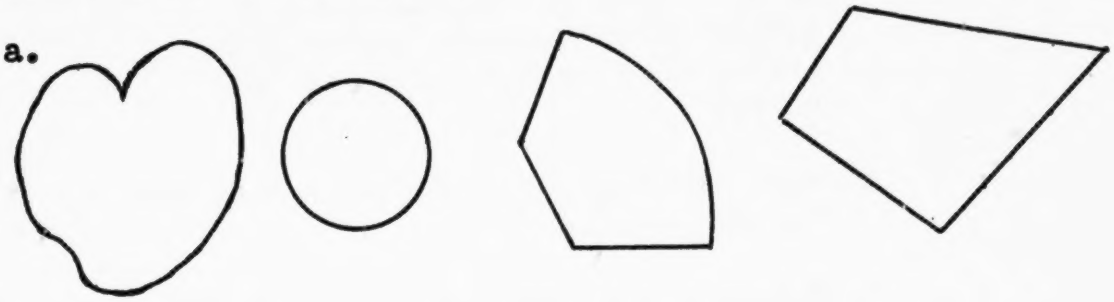
3. How are the curves in group *a* different from those in group *b*? (In group *b*, the curves were drawn by starting at a point, having the pencil point move so that it never left the paper, and coming back to the starting point.)

   Tell pupils that a curve which begins and ends at the same point is called a *closed curve*.

B. Meaning of a simple closed curve

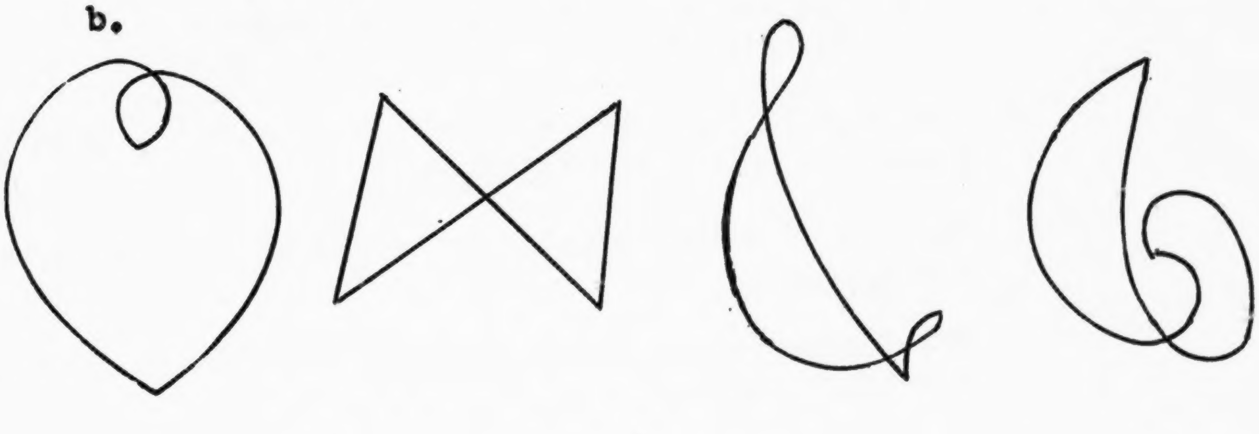
1. Have pupils draw some pictures of closed curves. Some may look like these:

a.



Others may look like these:

b.



2. How are the curves in group a different from those in group b?

   a. The curves in group a do not cross themselves. Closed curves that do not cross themselves are called simple closed curves.

   b. Refer to motivation.
   Which of the curves that you pictured are simple closed curves? Why?

   c. Tell pupils that because the geometric figures we have represented are all sets of points in a plane, we call them plane figures.

C. A simple closed curve separates the plane into three sets of points.

1. Draw a picture of a simple closed curve.

   a. Where are the points of the plane of the paper in relation to the simple closed curve? (inside the curve; outside the curve; on the curve)

   b. Into how many sets of points does the simple closed curve divide the plane? (3)

   c. What are the three sets of points?

   The three sets of points are:

   1) The set of points that form the geometric figure.
   2) The set of points inside the figure.
   3) The set of points outside the figure.

2. Tell pupils that the set of points inside the geometric figure is called the interior.

   The set of points outside the geometric figure is called the exterior.

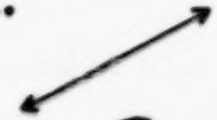
   The figure itself is the boundary between the interior and the exterior.

3. Have pupils draw a picture of a simple closed curve. Have them locate several points in the interior; several points in the exterior; several points on the boundary.

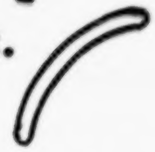
## II. Practice

A. Which are closed curves?
Which are simple closed curves?

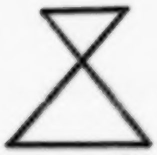
1.



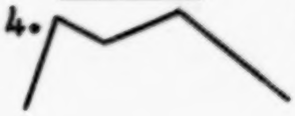
2.



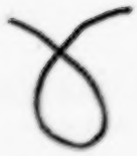
3.



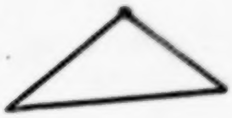
4.



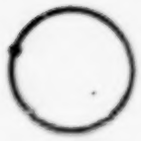
5.



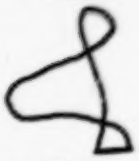
6.



7.



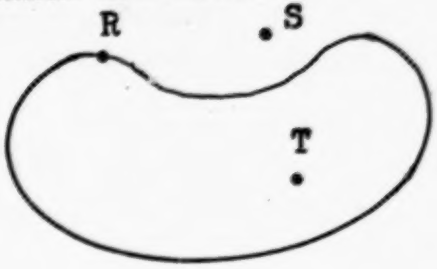
8.



B. The window frame in your classroom is a model of a simple closed curve.

Name three objects in your classroom which are models of simple closed curves.

C. The figure at the right is a picture of a simple closed curve and points R, S, T.



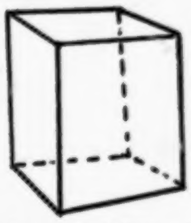
1. Which point is in the interior of the simple closed curve?

2. Which point is in the exterior of the curve?

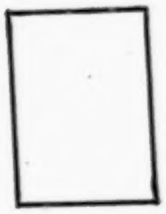
3. Which point is on the curve?

D. Which of the models of geometric figures pictured below are not plane figures? Explain.

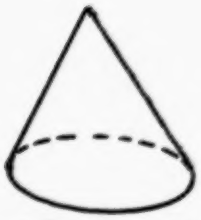
1.



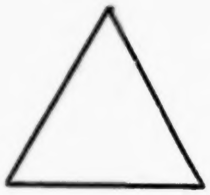
2.



3.

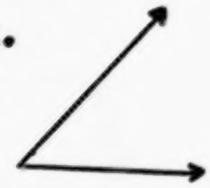


4.

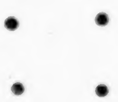


E. Which of the geometric figures pictured below are plane figures?

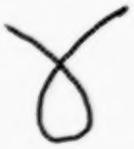
1.



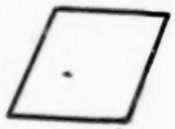
2.



3.



4.



F. If an insect crawls on the ceiling of a room, how can we describe the geometric figure represented by his path? (plane figure)

If the insect crawls from the ceiling to the wall, must his path now be a plane figure? Explain.

III. Summary

A. What is a closed curve?

B. What is the difference between a closed curve and a simple closed curve?

C. Describe the sets of points into which a simple closed curve separates the plane.

D. What is meant by a plane figure?

E. What mathematical vocabulary have you used today?

(closed curve, simple closed curve, interior, exterior, boundary, plane figure)

## Lesson 37

**Topic: Simple Closed Curves**

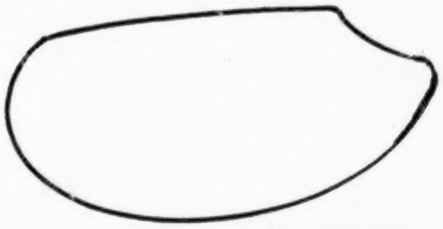
**Aim: To reinforce and extend the meaning of a polygon**

**Specific Objectives:**

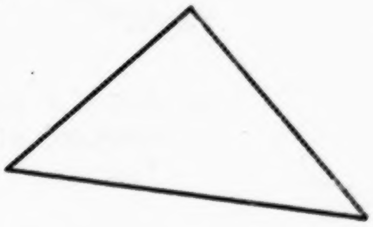
Meaning of polygon; naming a polygon
Classification of polygons
The circle as a special kind of simple closed curve

Challenge: Consider the following drawings:

a.



b.



In what way are they alike?
In what way do they differ?

I. Procedure

A. Meaning of polygon; naming a polygon

1. Refer to challenge.

a. Elicit that the figures are alike in that both are representations of simple closed curves. However, the simple closed curve in b is formed by the union of line segments.

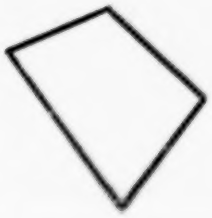
b. Tell pupils that a special name is given to a simple closed curve which is a union of line segments. We call it a polygon.

Each of the line segments is called a side of the polygon.

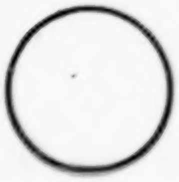
A common endpoint of two line segments is called a vertex of the polygon.

c. Which of the following represent polygons? Explain.

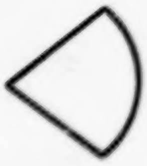
a.



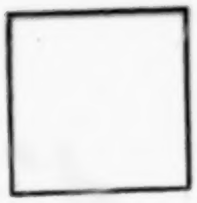
b.



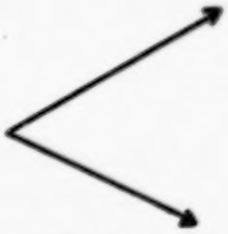
c.



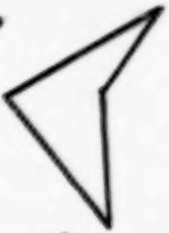
d.



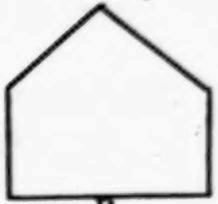
e.



f.

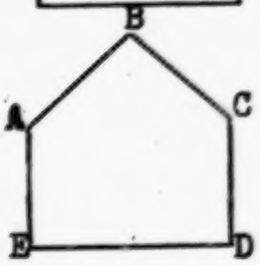


2. How can we name a polygon such as the one represented in the figure at the right?



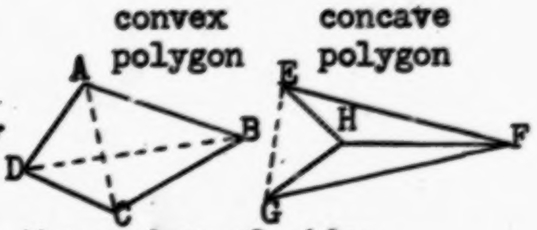
   a. Pupils may suggest naming each vertex of the polygon by a capital letter.

   b. Have pupils practice drawing pictures of polygons and naming them.



Polygon ABCDE

Note to Teacher: Polygons may be convex or concave. A polygon is convex if none of its diagonals are outside. It is concave if one or more diagonals are outside. Thus



In this course, convex polygons only will be considered.

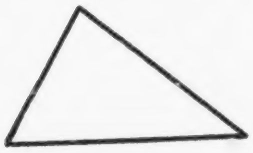
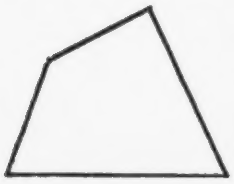
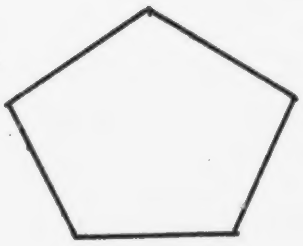
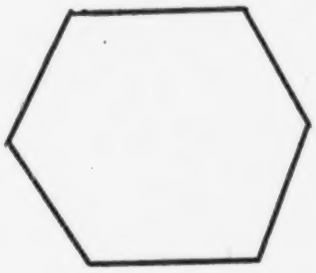
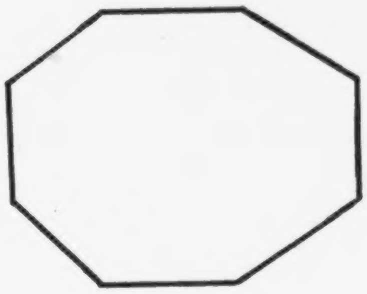
B. Classification of polygons according to the number of sides

1. Draw an example of a polygon that has:

   3 sides, 4 sides, ... 8 sides.

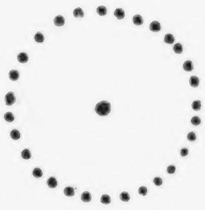
   Can you draw a polygon with only 2 sides.

2. Help pupil classify polygons according to the number of sides.

| Diagram | Number of Sides | Name |
|---|---|---|
|  | 3 | Triangle |
|  | 4 | Quadrilateral |
|  | 5 | Pentagon |
|  | 6 | Hexagon |
|  | 8 | Octagon |

C. The circle

1. Draw a point on a piece of paper and then draw another point which is a distance of one inch from the first point. Draw many points which are at a distance of one inch from the first point.



   a. What figure is suggested by your dots?

   b. Elicit that a circle is the set of all points in a plane which are the same distance from a point in the plane called the center.

2. Why is a circle a simple closed curve? What distinguishes a circle from a polygon?

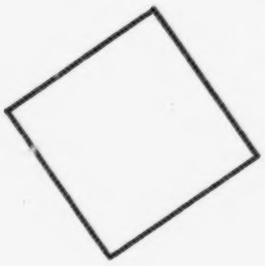
3. The rim of a drinking glass suggests a circle. Name five more objects in your classroom or in your home which suggest circles.

4. Have pupils draw pictures of circles freehand. Then have them use a compass to draw a picture of a circle. Why does the compass produce better results?

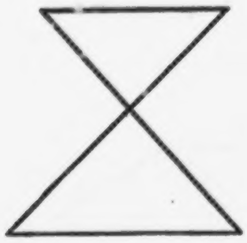
II. Practice

A. Which of the following represent polygons?

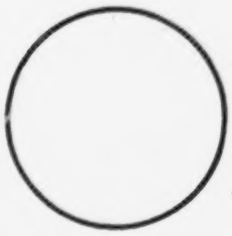
1.



2.



3.



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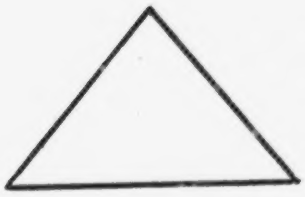
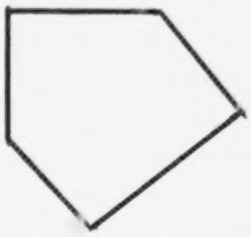
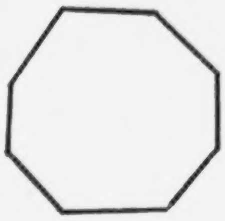


5.



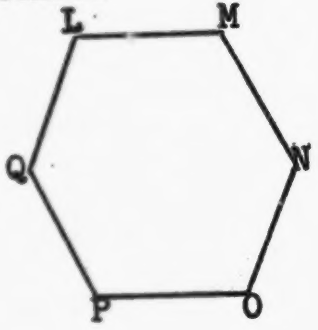
6.



B. Of the figures in A, which represent golygons, which represents a triangle? a quadrilateral? a pentagon? an octagon?

C. What special kind of simple closed curve is represented in A?

D. Consider the polygon represented at the right.



1. Name each vertex of the polygon.

2. Name the line segments which are the sides of the polygon.

3. Name the polygon.

4. What kind of polygon is this?

III. Summary

A. What is meant by a polygon?

B. How do we name a polygon?

C. How are polygons classified?

D. What do we call a polygon of three sides? four sides? five sides? six sides? seven sides? eight sides?

E. What do we call the set of all points in a plane which are the same distance from a fixed point in the plane?

F. What is the difference between a circle and a polygon?

## Lesson 38

Topic: Parallel Lines

Aim: To reinforce and extend the meaning of parallel lines

Specific Objectives:

Meaning of parallel lines
Through a point not on a given line there is no more than one line parallel to the given line
Special quadrilaterals: trapezoid, parallelogram

Challenge: Why will the two rails of a railroad track never intersect?

I. Procedure

A. Meaning of parallel lines

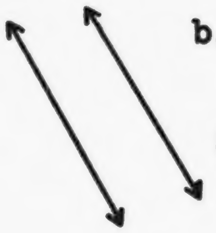
1. Discuss with pupils the need for the rails of a train track to accommodate the train wheels. Since these wheels are always the same distance apart, the rails must likewise remain the same distance apart.

2. Have pupils give other examples which suggest lines which do not meet.

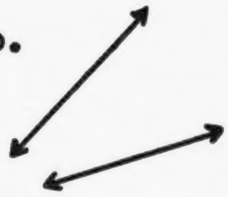
3. Tell pupils that lines in the same plane which do not intersect are called parallel lines. We use the symbol $\parallel$ for the words "is parallel to." Thus, $c \parallel d$ is read "line c is parallel to line d."

4. Which pairs of lines appear to be parallel?

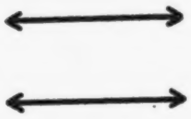
a.



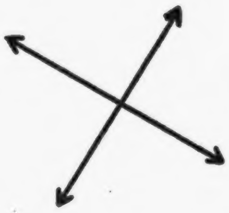
b.



c.



d.



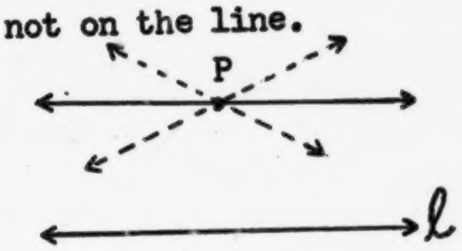
5. Consider the lines suggested by the intersection of the front wall and the ceiling, and the intersection of the side wall and the floor.

   a. Will these lines meet?
   b. Are the lines in the same plane?
   c. Are the lines parallel?

6. Tell pupils that two lines that do not intersect and do not lie in the same plane are called skew lines.

B. The number of lines that may be drawn parallel to a given line through a point not on the line

   1. Draw a picture of a line $\ell$ and mark a point P not on the line.



      a. Through point P draw a picture of a line parallel to line .

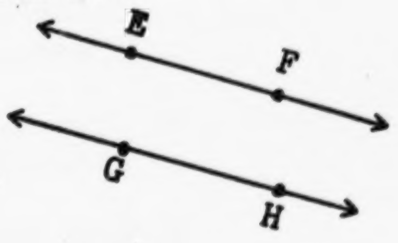
      b. Through point P can we draw another line, different from the first, parallel to line $\ell$?

      c. Through point P, how many lines parallel to line $\ell$ are there?

   2. Have pupils conclude that through a point not on a given line there is no more than one line parallel to the given line.

C. Special quadrilaterals

   1. Consider the parallel lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{GA}$ represented in the drawing at the right.

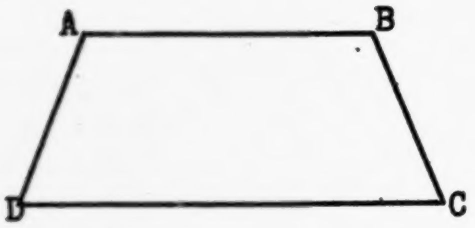


      a. What appears to be true of $\overline{EF}$ and $\overline{GH}$?

      b. Choose another segment on $\overleftrightarrow{EF}$ and one on $\overleftrightarrow{GH}$. What is true of these segments?

   2. Have pupils realize that if two lines are parallel, any segment of one line is parallel to any segment of the other line.

3. Consider the drawing at the right.



   a. How many sides are there in the polygon ABCD?

   b. What do we call a polygon which is the union of four line segments? (quadrilateral)

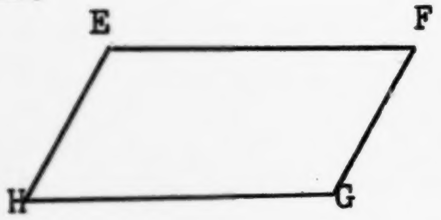
   c. What is true of $\overline{AB}$ and $\overline{DC}$? (parallel)

   d. Are $\overline{AD}$ and $\overline{BC}$ parallel?

4. Tell pupils that a quadrilateral which has only one pair of parallel sides is called a trapezoid.

5. Have pupils draw several pictures of trapezoids.

6. Examine the drawing at the right.



   a. How many sides are there in polygon EFGH?

   b. What do we call a polygon of 4 sides?

   c. What is true of $\overline{EF}$ and $\overline{HG}$? of $\overline{EH}$ and $\overline{FG}$?

7. Tell pupils that a quadrilateral which has both pairs of opposite sides parallel is called a parallelogram.

8. Have pupils draw several pictures of parallelograms.

II. Practice

A. What appears to be true of the two lines suggested by the intersection of the front wall and the ceiling, and the intersection of the front wall and the floor?

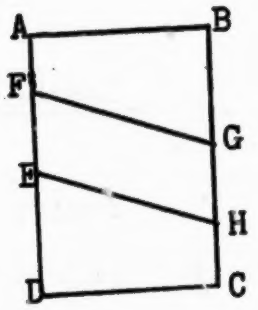
B. Give three examples of pairs of parallel lines suggested by objects in your classroom.

C. Give three examples of pairs of skew lines suggested by objects in your classroom.

D. If line m is parallel to line n, and line n is parallel to line r, what do you think is true about lines m and r?

E. Draw a picture of a four-sided plane figure with no two sides parallel; with one pair of sides parallel; with both pairs of opposite sides parallel.

F. What do you call each of the figures you drew for E?



G. Consider the drawing at the right.

1. What special quadrilateral is ABGF? Why?
2. What special quadrilateral is EHCD? Why?
3. What special quadrilateral is EFGH? Why?

H. Is every rectangle a parallelogram? Explain.
Is every parallelogram a rectangle? Explain.

III. Summary

A. How may two lines be related to each other? (intersect, parallel, skew)

B. What is the difference between parallel lines and skew lines?

C. What is the greatest number of lines that may be drawn parallel to a given line through a given point?

D. In what way is a trapezoid like all quadrilaterals? In what way is it special?

E. In what way is a parallelogram like all quadrilaterals? In what way is it special?

F. What new vocabulary have you learned today? (skew, trapezoid, parallelogram)

## Lessons 39 and 40

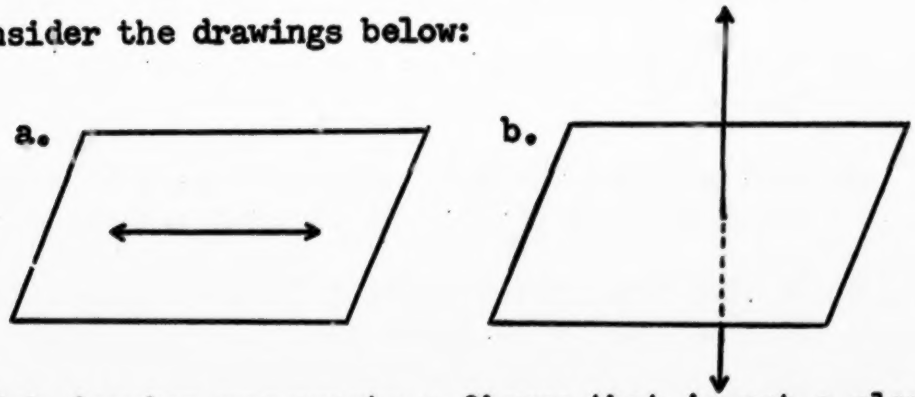
Topic: Space Figures

Aim: To develop an understanding of space figures

Specific Objectives:

Meaning of space figures
Plane drawings picture space figures
Recognition of some common space figures: prism (rectangular, triangular), sphere, cylinder, cone, pyramid

Challenge: Consider the drawings below:



Which drawing represents a figure that is not a plane figure?

I. Procedure

A. Meaning of space figure

1. Elicit that the drawing in a represents a plane figure because it pictures points, all of which lie entirely within a plane.

   The drawing in b does not represent a plane figure, since not all the points lie in one plane.

2. Discuss with pupils that even though the set of points pictured in b do not lie in one plane, they are a subset of the set of points of space.

   Tell pupils that such figures are called space figures rather than plane figures.

3. Have pupils give several examples of objects which suggest space figures which are not plane figures. (a ball, a book, a desk)

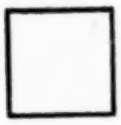
B. Picturing a space figure

1. Have pupils make a drawing of: a shoebox, a baby block, an orange, an ice cream cone, a juice can.

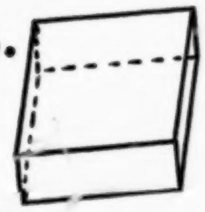
2. Elicit that even though these are examples of space figures, we have used drawings on a plane to represent them.

3. Which are drawings of space figures that are not plane figures?

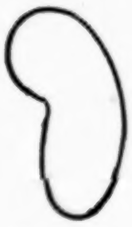
a.



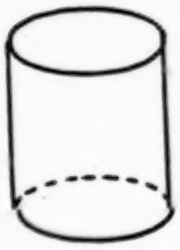
b.



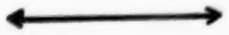
c.



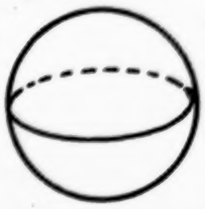
d.



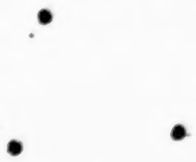
e.



f.

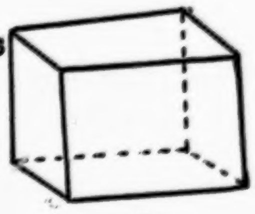


g.

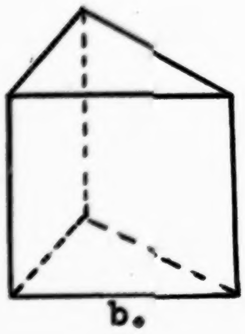


C. Recognition of some common space figures

1. Have pupils consider the drawings at the right.



a.



b.

a. Tell pupils that each drawing represents a space figure called a prism.

b. Consider figure a. What familiar objects are models of such a prism? (shoe box, tooth paste carton, a book)

1) Any one of the flat surfaces of the prism is part of a plane and is called a face of the prism. How many faces are there in this prism?

2) The "top" and "bottom" faces of the prism are called bases of the prism.

3) A prism all of whose faces represent rectangular regions is called a rectangular prism.

c. Pose questions such as the following with reference to figure b.

   1) How many faces are there in this prism?

   2) What kind of region are its bases? (triangular region)

   3) What kind of region is each of the other faces of this prism? (rectangular)

   4) Because the bases of this prism are triangular regions, what do you think we should call this kind of prism? (triangular prism)

   5) Name some objects which suggest a triangular prism. (cheese wedge, glass prism used to show the diffraction of light)

2. Using similar procedures have pupils learn to recognize the following space figures:

| | |
|---|---|
| sphere | cone |
| cylinder | pyramid |

3. When describing and naming space figures, we often use a "double" name, as for example, rectangular prism, square pyramid, and so on.

   The "first" name refers to the base, and the "second" name refers to the faces.

4. Have pupils practice making drawings (perhaps actual models) to represent the space figures studied.

5. Have pupils realize that just as a simple closed curve divides the plane in which it lies into three sets of points, so a prism divides space into three subsets:

   a. the set of points on the surface of the prism

   b. the set of points outside the prism, the exterior

   c. the set of points inside the prism, the interior.

## II. Practice

A. Which of the following suggest space figures that do not lie in a plane?

1. a ray
2. a brick
3. an orange
4. a table top
5. a line segment
6. a triangle
7. a funnel
8. a swimming pool
9. a circle
10. an Indian tepee

B. Some of these objects suggest rectangular prisms, cylinders, pyramids, spheres or cones. Classify each.

1. basket ball
2. water pipe
3. funnel
4. snare drum
5. grapefruit
6. lump of sugar
7. coffee can
8. hat box
9. book
10. tent pegged at four corners

C. Make a drawing of the geometric figure that each of the objects in B suggests.

D. Name three objects which are shaped like a rectangular prism; like a cube; like a cylinder; like a cone; like a sphere.

E. Which of these is concerned with the surface of the model, and which is concerned with the interior of the surface?

1. peeling a banana
2. eating the banana after it has been peeled
3. filling a bathtub with water
4. painting the walls of a room
5. sipping soda through a straw
6. putting a gift in a box
7. wrapping a box

F. Substitute the name of the common object.

1. The cookies were placed in a rectangular solid.
2. He hit the sphere over the fence.
3. Some Indians still live in cones.
4. Campers sometimes sleep in pyramids.
5. Frozen orange juice comes in cylinders.

G. Bring in pictures that show objects, buildings, etc., which suggest each of the geometric forms studied.

III. Summary

A. What is the difference between the geometric figure suggested by a page of a textbook and that suggested by the book itself?

B. In which space figure do you find the following as a face or as a base? Explain.

1. circular region

2. rectangular region

3. square region

4. triangular region

C. Which of the space figures you have studied does not have a plane figure as a face or base?

D. What new vocabulary have you learned today?

(space figure, face, base, edge, rectangular prism, triangular prism, cube, cylinder, cone, pyramid)

# CHAPTER IV

This chapter contains suggested procedures for helping pupils understand and use the concepts of:

set intersection
Venn **diagrams**
divisibility
prime number
composite number
greatest common factor
relatively prime numbers

In working with rational numbers, the pupil often needs to change the form of the numeral that names the number. Factoring is an important part of this process and is therefore presented at this time. For example, it is used in "reducing a fraction to its lowest terms" as follows:

$$\frac{8}{12} = \frac{4 \times 2}{4 \times 3} = \frac{4}{4} \times \frac{2}{3} = 1 \times \frac{2}{3} = \frac{2}{3}$$

Any positive integer n, greater than 1, is called a prime number if it is divisible only by 1 and n. If an integer n, greater than 1, is not a prime, it is said to be a composite number.

An integer n can be factored into a product of prime numbers in one and only one way. This fact is often referred to as the unique factorization theorem. It is extremely useful in determining the greatest integral factor of both of two integers, and in finding least common denominators.

The discussions of prime and composite numbers exclude the number one from the possibility of being either prime or composite. It is important that the number 1 not be prime, for otherwise the unique factorization theorem would not hold true. If one were classified as a prime, we could consider 2 x 3 or 1 x 2 x 3, or 1 x 1 x 2 x 3, etc. as prime factorizations of 6. Thus, the factorization possibilities would be endless.

The concept of greatest common factor, presented in this chapter will be utilized frequently in succeeding chapters as pupils work with rational numbers. If the greatest common factor of two numbers is 1, then the numbers are said to be relatively prime, or prime to each other. Lest this terminology cause some pupils to think that both of the numbers must themselves be prime numbers, it is important to point out that one or both of the numbers may be composite. For example, in $\frac{8}{15}$, the greatest common factor of 8 and 15 is 1; therefore 8 and 15 are relatively prime numbers. However, 8 and 15 are themselves composite numbers, not prime numbers.

# CHAPTER IV

## FACTORING

## Lessons 41-44

### Lesson 41

Topic: Intersection of Sets

Aim: To learn the meaning of the operation of intersection of sets

Specific Objectives:

Meaning of set intersection; symbol for intersection
Using diagrams to represent sets and intersection of sets (Venn diagrams)

Challenge: Consider the set of pupils in the class who have blue eyes:

E = {Robert, Laura, Nita, Harold}

Consider the set of pupils in the class who are 5'6" tall or over:

H = {Nita, Paul, Leonard, Harold, Sam}

Who are the members of the set of blue-eyed pupils who are 5'6" tall, or over?

I. Procedure

A. Meaning of set intersection; symbol for intersection

1. Elicit that Nita and Harold are the only pupils whose names appear as members of both sets. That is to say, they are the only members of the set of pupils who are both blue-eyed and 5'6" tall, or over.

   {Nita, Harold} is the set of all members common to sets E and H.

2. Give the set of elements common to the following pairs of sets:

   a. {Jane, Sue, Mary}; {Ann, Lynn, Mary}

   b. {5,6,8,10,11}; {1,2,5,7,8,9}

   c. {0,2,4,6,8,10}; {0,4,8,12}

   d. {a,e,i,o,u}; {a,b,c,d,e}

   e. {3,6,9,12,15,18}; {4,8,12,16,20}

3. Tell pupils that the set which consists of only those elements which are common to a pair of sets is called the intersection of the pair of sets.

4. Tell pupils that the symbol for intersection is $\cap$.

   $A \cap B$ means the intersection of set A and set B.

   We read this: A intersection B or the intersection of set A and set B.

   $C = A \cap B$ means set C is the intersection of set A and set B.

5. Consider the sets: $R = \{1,2,3\}$ and $S = \{4,6,8,10,12\}$

   a. Which elements have these sets in common?

   b. How do we describe sets which have no elements in common? (disjoint)

   c. Elicit that the intersection of disjoint sets is the empty set.

      $\{1,2,3\} \cap \{4,6,8,10,12\} = \emptyset$

6. List the members of the intersection of the following pairs of sets.

   a. $\{1,10,100,1000\} \cap \{10,20,30,40\}$

   b. $\{0,1,2,3,4,5,\ldots\} \cap \{1,3,5,7,\ldots\}$

   c. $\{2,4\} \cap \{6,8,10\}$

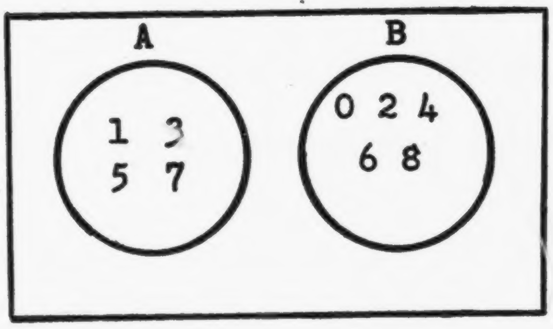
   d. $\{0\} \cap \{0,5,10\}$

B. Using diagrams to represent sets and intersection of sets

1. Tell pupils that diagrams, called Venn diagrams, are often used to represent sets and relationships between sets.

   a. Consider $A = \{1,3,5,7\}$ and $B = \{0,2,4,6,8\}$.

1) Let us picture the members of set A inside the circle labeled A, and the members of set B inside the circle labeled B.

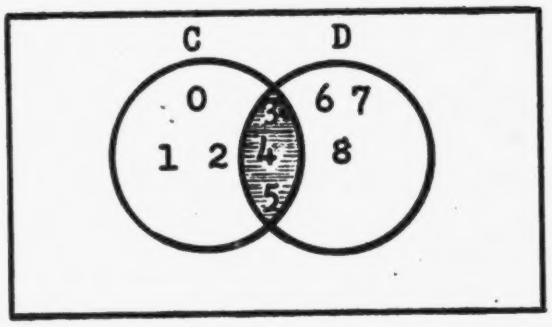


2) Are any of the same elements in both sets? How are sets A and B related? (They are disjoint sets.)

b. Consider C = {0,1,2,3,4,5} and D={3,4,5,6,7,8}

1) Are these sets disjoint? What elements are members of both sets?

2) Have pupils see that sets C and D may be pictured as follows:



What is C ∩ D? ({3,4,5})

The shaded portion of the drawing represents the intersection of set C and set D.

Note to teacher: In the theory of sets, the rectangular region of the Venn diagram represents what is called the universe. The universe is the total set of elements about which the diagram is concerned.

2. Have pupils draw diagrams of the pairs of sets in A-6, showing the members of the intersection set in each case.

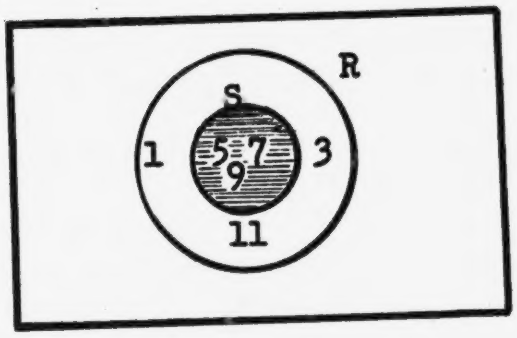
3. How can we use diagrams to picture a subset of a set?

a. Consider R = {1,3,5,7,9,11} and S = {5,7,9}

1) How are these sets related? (S is a subset of R.)

2) What is R ∩ S? ({5,7,9})

3) If the elements of set R are pictured within circle R, where would you place the circle containing the elements of set S? (inside circle R)



b. Tell how the second set in each of the following pairs of sets is related to the first.

1) L = {0,2,3,8,11,13}; M = {11,13}

2) P = {10,20,30,40,50,60}; Q = {10,60}

3) H = {0,5,10,15}; K = {0}

c. In b above, what is L ∩ M?; P ∩ Q?; H ∩ K?

d. Draw diagrams of the pairs of sets in b above.

II. Practice

A. List the members of the intersection of the following pairs of sets.

1. F = {father, mother, Lilly, Jack}; H = {Lou, Peter, Jack}

   F ∩ H = {Jack}

2. C = {m,n,p,y}; D = {a,m,e,n,i}

3. F = {2,4,6,8,10,12}; L = {3,6,9,12}

4. N = {4,8,12,24}; P {3,6,9,18}

5. The set of whole numbers; the set of odd numbers.

6. $\{\frac{1}{4}, \frac{2}{3}, \frac{5}{8}\}$; $\{\frac{1}{2}, \frac{5}{8}, \frac{6}{11}\}$

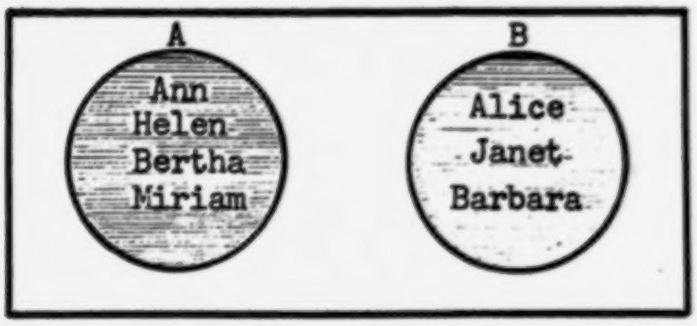
B. Make up pairs of sets whose intersection is:

1. two or more elements
2. one element
3. no elements

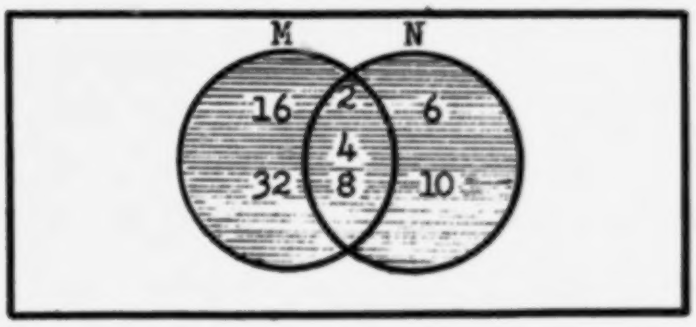
C. Draw diagrams of the pairs of sets in A showing the members of the intersection set in each case.

D. If A = {Ann, Helen, Bertha, Miriam} and B = {Alice, Janet, Barbara}, what is A ∪ B?

How is this shown in the drawing below?



E. If M = {2,4,8,16,32} and N = {2,4,6,8,10}, what is M ∪ N? How is this shown in the drawing below?



III. Summary

A. What is meant by the intersection of two sets? What is the symbol for intersection?

B. What is the intersection set of two disjoint sets?

C. How can we use drawings to represent the following?

1. a pair of disjoint sets
2. a pair of sets which have some elements in common
3. the intersection of a pair of sets which are not disjoint
4. a pair of sets in which one is a subset of the other

D. What new vocabulary have you learned today?

(intersection)

E. What new symbol have you learned today? ( $\cap$ )

## Lesson 42

Topic: Factoring

Aim: To understand the concept of prime number

Specific Objectives:

Review of factors of a product
Meaning of divisibility
Meaning of prime number; composite number

Challenge: In how many ways can you express 6 as a product of two whole numbers?

I. Procedure

A. Factors of a product

1. Elicit that 6 can be expressed as a product of two whole numbers in two ways:

$$6 = 2 \times 3$$
$$6 = 6 \times 1$$

2. Have pupils recall that a factor is a number used in multiplication to form a product. Thus, 2, 3, 6, and 1 are said to be factors of 6.

a. Express 12 as a product of factors.

$$12 = 2 \times 6$$
$$12 = 3 \times 4$$
$$12 = 12 \times 1$$

What is the set of all factors of 12? ({1,2,3,4,6})

b. Express 19 as a product of factors: 19 = 1 x 19.
What is the set of all factors of 19? ({1,19})

3. Elicit that any whole number may be expressed as the product of factors.

Note to teacher: Although fractional numbers can be factors of a product, we shall limit ourselves to the set of whole numbers.

4. List the set of all the factors of each number named.

   a. 20

   b. 36

   c. 37

   d. 42

   e. 66

B. Divisibility

1. Have pupils compute the following:

| a. | b. |
|---|---|
| 6 ÷ 2 | 6 ÷ 5 |
| 6 ÷ 3 | 12 ÷ 7 |
| 12 ÷ 4 | 8 ÷ 3 |
| 15 ÷ 5 | 15 ÷ 2 |

   How are the results of the computations in group a different from those in group b? (Each computation in group a results in a zero remainder, whereas those in group b have non-zero remainders.)

2. Tell the pupils a number is said to be divisible by another number only if there is a zero remainder. 6 is divisible by 2 because 2 divides 6 with a zero remainder.

   Is 10 divisible by 2? by 5? by 3? Explain.

3. Have pupils consider the following:

   a. Is 6 divisible by 2? Is 2 a factor of 6?
   b. Is 6 divisible by 3? Is 3 a factor of 6?
   c. Is 12 divisible by 4? Is 4 a factor of 12?
   d. Is 15 divisible by 5? Is 5 a factor of 15?

4. Elicit that if a number is divisible by a second number, the second number is a factor or divisor of the first number.

5. Have pupils test to see if the second number named in each pair is a factor of the first.

   a. 128 ÷ 4     b. 308 ÷ 14     c. 528 ÷ 21     d. 57,510 ÷ 135

6. Fill in the table:

| Product | One Factor (Divisor) | Other Factor |
|---|---|---|
| 10 | 2 | ? |
| 15 | 5 | ? |
| 96 | 12 | ? |

Elicit that given a product and one factor, the other factor can be found by dividing the product by the given factor.

7. Have pupils use the following systematic procedure for finding the set of all factors of a number.

What is the set of all factors of 20?

$20 \div 1 = 20$ This gives us the pair of factors: 1,20

$20 \div 2 = 10$ This gives us the pair of factors: 2,10

$20 \div 3 = ?$ No factors

$20 \div 4 = 5$ This gives us the pair of factors: 4,5

$20 \div 5 = 4$ We already have this pair: 4,5

We don't have to try any more divisions. Why?

The set of all the factors of 20 is $\{1,2,4,5,10,20\}$

C. Prime number; composite number

1. List the set of all factors of the counting numbers named below:

a. 2
b. 3
c. 7
d. 10
e. 13
f. 15

Which of the above have exactly two factors? (2,3,7,13)

2. Tell pupils that a counting number which has exactly two factors is called a prime number.

a. 2 is a prime number because its only factors are 2 and 1.

b. 3, 5, and 7 are prime numbers. Why?

c. Why is 4 not a prime number? (It has three factors: 1,2,4.)

d. Why would you not expect any even number except 2 to be a prime number?

e. Why are the counting numbers 6, 8, and 9 not prime?

f. How many factors has the counting number 1? Elicit that 1 is the only counting number which has exactly one factor.

Note: Since the number 1 has only one factor, it is not considered to be a prime number.

g. What is the least prime number?

3. Tell which of the numbers named below are prime.

a. 16 b. 23 c. 35 d. 29

4. Tell pupils that any counting number other than 1 that is not prime is called a composite number.

5. Which of the numbers named below are composite?

a. 18 b. 21 c. 32 d. 31

II. Practice

A. List the five least counting numbers which have 7 as one of their factors.

B. Test to see if the second number named in each pair is a factor of the first.

1. 369; 3
2. 432; 18
3. 1217; 27
4. 39184; 124

C. Find the set of all factors of the numbers named below.

1. 18 2. 41 3. 64 4. 70

D. Are the numbers named below prime or composite? Justify your answer.

1. 5 2. 8 3. 19 4. 22

5. 58 6. 67 7. 95

E. What is the next prime number after 79?
What is the intersection of the set of prime numbers and the set of odd numbers less than 15?

F. OPTIONAL ( Sieve of Eratosthenes)
Use the following procedure discovered by the Greek scholar, Eratosthenes, to divide all the counting numbers except 1 into two sets: the set of composite numbers and the set of prime numbers. We shall limit ourselves to the counting numbers through 100.

1. Copy this arrangement of numerals for counting numbers through 100, except 1.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 11 | 21 | 31 | 41 | 51 | 61 | 71 | 81 | 91 |
| 2 | 12 | 22 | 32 | 42 | 52 | 62 | 72 | 82 | 92 |
| 3 | 13 | 23 | 33 | 43 | 53 | 63 | 73 | 83 | 93 |
| 4 | 14 | 24 | 34 | 44 | 54 | 64 | 74 | 84 | 94 |
| 5 | 15 | 25 | 35 | 45 | 55 | 65 | 75 | 85 | 95 |
| 6 | 16 | 26 | 36 | 46 | 56 | 66 | 76 | 86 | 96 |
| 7 | 17 | 27 | 37 | 47 | 57 | 67 | 77 | 87 | 97 |
| 8 | 18 | 28 | 38 | 48 | 58 | 68 | 78 | 88 | 98 |
| 9 | 19 | 29 | 39 | 49 | 59 | 69 | 79 | 89 | 99 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

2. Begin with 2; cross out all the numerals for multiples of 2, except itself.

3. Find the next numeral not crossed out; it is 3.
Cross out all the numerals for multiples of 3, except itself.

4. Find the next numeral not crossed out; it is 5. Cross out all the numerals for multiples of 5, except itself.

5. Continue this process for all the numerals in the arrangement.

6. The numerals that have not been crossed out are the names of the prime numbers less than 100.
What do the numerals that have been crossed out represent?

## III. Summary

A. What do we call the numbers which are multiplied to form a product?

B. If one number is divisible by a second number, what can we say about the second number?

C. If a product and one factor are known, how can we find the other factor?

D. How many factors does a prime number have? What are the factors?

E. What is the meaning of a composite number?

F. Is there a number that has no factors? Explain.

G. How many numbers have only 1 factor?

H. Is there a number which is a factor of all numbers?

I. What new vocabulary have you learned today?

(factor, divisible, prime number, composite number)

## Lesson 43

Topic: Composite Numbers

Aim: To understand how composite numbers may be expressed as the product of prime factors

Specific Objectives:

Prime factors
Complete factorization
Unique factorization property

OPTIONAL: Tests for divisibility

Challenge: Can every composite number be given as a product using only prime numbers?

I. Procedure

A. Prime factors

1. What are all the possible pairs of factors of the number 12?

$12 = 1 \times 12$
$12 = 2 \times 6$
$12 = 3 \times 4$

a. Elicit that the number 2 is a prime number; also, 2 is a factor of 12. We say, therefore, that 2 is a prime factor of 12.

b. Does 12 have another prime factor? What is it?

c. The set of all prime factors of 12 is $\{2,3\}$.

2. What are two ways of expressing 15 as a product of pairs of factors?

$15 = 1 \times 15$
$15 = 3 \times 5$

a. What are the prime factors of 15?

b. What is the set of all prime factors of 15?

3. Find the set of all prime factors of each composite number named below.

   a. 18 b. 21 c. 36 d. 45

B. Complete factorization

1. Is it possible to express any composite number as the product of prime factors? (challenge)

   a. Consider 30. Is it prime or composite? Why?

      1) Express 30 as a pair of factors, neither of which is 1. For example, 30 = 3 x 10.

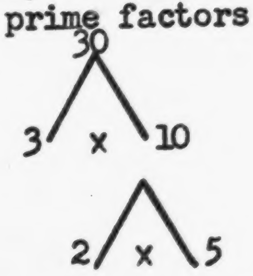
      2) Is the factor 3 prime or composite? (prime)
      Is the factor 10 prime or composite? (composite)

      3) Since 10 is not a prime number, find a pair of factors of 10. (2 and 5)

      4) How can we now express 30 as the product of prime factors?

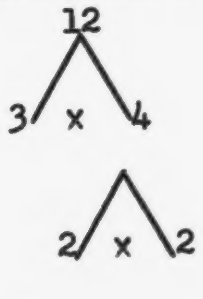
      30 = 3x2x5

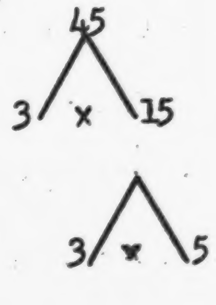
      5) A factor tree, such as that below, is helpful in finding the prime factors of a number.



The prime factors of 30 are 3, 2, and 5. 30 = 3x2x5

   b. Have pupils explain how a factor tree can be used to find the prime factors of 12; of 45.





12 = 2x2x3 or $2^2$ x 3

45 = 3x3x5 or $3^2$ x 5

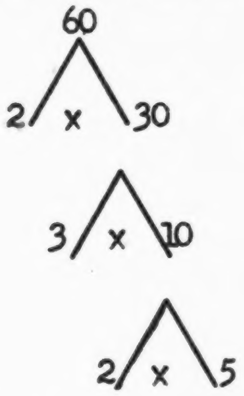
2. Tell pupils that when a composite number is expressed as a product of prime numbers, we say that it is factored completely. We have found the prime factorization of the of the number.

3. Factor completely: 26 36 60 72

C. Unique factorization property

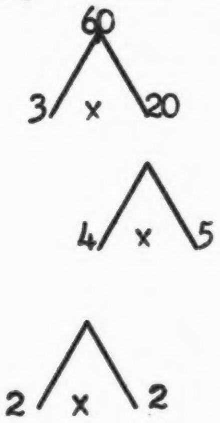
1. Have pupils find the complete factorization of the number 60.

   a. Some pupils may factor as follows:



   Elicit that this is a complete factorization of 60.

   b. Other pupils may factor as follows:



   Elicit that this is a complete factorization of 60.

   c. Does $2 \times 3 \times 2 \times 5 = 3 \times 5 \times 2 \times 2$? Explain.

2. After several such illustrations, have pupils conclude that if we disregard order, a composite number can be factored into a product of prime numbers in only one way. (Unique Factorization Theorem)

D. (OPTIONAL) Tests for divisibility: by 3; by 9

1. Guide pupils to see that it is often possible to tell whether a number is a factor of another number without dividing the second number by the first.

a. Elicit that we can tell that a number is divisible by 2 without dividing, if the final digit of the number is 0, 2, 4, 6, or 8.

b. We can tell that a number is divisible by 5 without dividing, if the final digit of the number is 0 or 5.

2. How can we tell whether a number is divisible by 3 without dividing?

   a. Have pupils observe the pattern in the following table.

| Number | Sum of Numbers Named by the Digits |
|---|---|
| 3 | 3 |
| 6 | 6 |
| 9 | 9 |
| 12 | 1+2=3 |
| 15 | 1+5=6 |
| 18 | 1+8=9 |
| 21 | 2+1=3 |
| 24 | 2+4=6 |
| 27 | 2+7=9 |
| 30 | 3+0=3 |
| 33 | 3+3=6 |
| 36 | 3+6=9 |

   b. Elicit that the number in the first column is divisible by 3; the number in the second column is divisible by 3.

   c. Have pupils try to find a number divisble by 3 in which the sum of the numbers named by the digits is not divisible by 3. (They cannot.)

   d. Have pupils conclude that a number is divisible by 3, if and only if, the sum of the numbers named by the digits is divisible by 3.

   e. Which of the numbers named below are divisible by 3?

      1) 306
      2) 432
      3) 5616
      4) 9355
      5) 204684

3. List the first twelve counting numbers which are divisible by 9.

   a. Compute the sum of the numbers named by the digits for each such number listed.

| Number -- | Sum of Numbers Named by the Digits |
|---|---|
| 9 | 9 |
| 18 | 1+8=9 |
| 27 | 2+7=9 |
| 36 | 3+6=9, and so on |
| . | |
| . | |
| . | |

   b. Are the sums divisible by 9?

   c. Can you find a number divisible by 9 in which the sum of the numbers named by the digits is not divisible by 9?

   d. Have pupils conclude that a number is divisible by 9, if and only if, the sum of the numbers named by the digits is divisible by 9.

   e. Which of the numbers named below are divisible by 9?

      1) 639
      2) 4788
      3) 51629
      4) 20043
      5) 817263

II. Practice

A. Which of the numbers named below have been completely factored? Explain.

1. 24 = 2x12
2. 60 = 4x3x5
3. 56 = 2x2x2x7

B. Use factor trees to rename the following as products of prime numbers.

1. 36
2. 42
3. 50
4. 120
5. 380

C. Find the set of prime factors of each of these numbers.

1. 22
2. 27
3. 40
4. 130
5. 630

D. Completely factor each number named below.

1. 54
2. 96
3. 390
4. 480
5. 2000

E. Give the complete factorization in exponential form.

1. 18
2. 75
3. 96
4. 360
5. 500

F. The following are the complete factorization of some numbers in exponential form. What are the numbers?

1. $2^2 \times 3 = \square$
2. $3 \times 2^3 = \square$
3. $3^2 \times 7 = \square$
4. $3^3 \times 5^2 = \square$

III. Summary

A. What is meant by a prime factor of a number?

B. How can you tell when a number has been factored completely?

C. If order is disregarded, in how many ways can you name a number as a product of prime factors?

## Lesson 44

**Topic:** Greatest Common Factor

**Aim:** To develop an understanding of the meaning of the greatest common factor of two (or more) numbers.

**Specific Objectives:**

Meaning of common factor
Finding the greatest common factor
Relatively prime numbers

**Challenge:** What is the set of all factors of 12?
What is the set of all factors of 30?
What is the intersection of these two sets?

### I. Procedure

#### A. Meaning of common factor

1. Elicit that:

   the set of all factors of 12 is $\{1,2,3,4,6,12\}$;
   the set of all factors of 30 is $\{1,2,3,5,6,10,15,30\}$;
   $\{1,2,3,4,6,12\} \cap \{1,2,3,5,6,10,15,30\}=\{1,2,3,6\}$

2. What is the set of all common factors of 12 and 30? Elicit that the intersection of the set of all factors of 12 and the set of all factors of 30, is the set of all common factors of 12 and 30.

3. Have pupils list the members of the following sets:

   a. the set of all factors of 16; of 24

   b. the set of all common factors of 16 and 24

   c. the intersection of the set of all factors of 16 and the set of all factors of 24

   d. Elicit that sets described in b and c are the same.

4. After several such illustrations, have pupils conclude that the intersection of the set of all factors of one number and the set of all factors of a second number is the set of all common factors of the two numbers.

B. Finding the greatest common factor (G.C.F.) of two (or more) numbers.

1. Consider the numbers 28 and 42.

    a. What is the set of factors of each of these numbers?

        1) The set of factors of 28 is $\{1,2,4,7,14,28\}$.

        2) The set of factors of 42 is $\{1,2,3,6,7,14,21,42\}$.

    b. What is the intersection of these sets of factors?

        $\{1,2,4,7,14,28\} \cap \{1,2,3,6,7,14,21,28\}=\{1,2,7,14\}$

    c. What is the greatest number in the intersection set? (14)

    d. Tell pupils that we call 14 the greatest common factor of 28 and 42.

2. Have pupils find the greatest common factor of each pair of numbers named below.

    a. 6 and 8 b. 24 and 30 c. 30 and 75 d. 27 and 63

3. Have pupils use prime factorization in finding the greatest common factor.

    a. Find the greatest common factor of 24 and 30.

        1) Have pupils express 24 as the product of prime factors; 30 as the product of prime factors.

            24 = 2x2x2x3

            30 = 2x3x5

        2) What prime factors do these numbers have in common? (2 and 3)

        3) Elicit that the product of the common prime factors, 2x3, is the greatest common factor.

    b. Find the greatest common factor of 48 and 60.

1) 48 = 2x2x2x2x3
   60 = 2x2x3x5

2) Elicit that the product 2x2x3, or 12, is the greatest common factor of 48 and 60.

4. Have pupils use prime factorization to find the greatest common factor of the following pairs of numbers.

   a. 12 and 28  b. 10 and 35  c. 42 and 60

C. Relatively prime numbers

1. Have pupils find the greatest common factor of the following pairs of numbers:

   a. 3 and 4  b. 6 and 7  c. 9 and 13  d. 25 and 27

2. Elicit that in each case, the greatest common factor is 1.

3. Tell pupils that when the greatest common factor of two numbers is 1, the numbers are said to be relatively prime.

II. Practice

A. List the members of the set of all factors of each number named below.

   1. 14; 42  2. 8; 36  3. 27; 72  4. 35; 90

B. List the members of the set of all common factors for each pair of numbers named in A.

C. Let set A contain as members all the factors of 16; let set B contain as members all the factors of 24.

   1. Draw a Venn diagram for set A and set B.

   2. Does $A \cup B$ or $A \cap B$ contain only the common factors of 16 and 24?

D. Find the greatest common factor of each pair of numbers named below.

   1. 16 and 30  2. 8 and 11  3. 32 and 80  4. 15 and 22
   5. 38 and 57

E. Which pairs of numbers in D are relatively prime?

III. Summary

A. What is meant by a common factor of two numbers?

B. How do you find the members of the set of all common factors of two numbers?

C. What is meant by the greatest common factor of two numbers?

D. When are two numbers relatively prime?

E. What new vocabulary have you learned today? (common factor, greatest common factor, relatively prime).

# CHAPTER V

This chapter contains suggested procedures for helping pupils develop concepts of:

rational number
other names for rational numbers
the simplest name for a rational number
rational numbers and the number line
density and order in the set of rational numbers
operations with rational numbers (multiplication and division)
properties of operations with rational numbers

At this stage in the pupil's development of mathematical understanding, we are concerned with only the non-negative rational numbers. In defining a rational number as the quotient of two whole numbers (excluding division by 0), we are using a definition that will eventually be extended to include the quotient of two integers with the second integer not 0.

Throughout the chapter, emphasis is placed upon the principle that numbers have many names and the name we select at a certain time for a particular number is one that suits our purpose. Thus, since a fraction is a name for a number, pupils are led to see that many fractions name the same rational number. The use of the number one and various names for one is stressed in relation to changing the names of numbers represented by fractions.

The fact that every rational number is associated with a point on the number line is emphasized along with the exploration of concepts of order and density of the set of rational numbers.

The operation of multiplication of numbers named by fractions is the easiest operation to perform over the set of rational numbers. Essentially, the operation is based on the definition of multiplication, which states that to find the product of two given rational numbers, we multiply the numerators and write a numeral for this product over a numeral naming the product of the denominators. However, the ease with which multiplication of rational numbers can be performed is not the primary reason for presenting multiplication before addition (or subtraction). To add (or subtract) rational numbers, the terms of the fractions which name these numbers must often be changed. To change these terms it is necessary to know how to multiply rational numbers.

Division of rational numbers is approached through illustrations that indicate dividing by a number is the same as multiplying by its reciprocal, or multiplicative inverse.

The lessons in this chapter (and in the next) provide for investigation by pupils of the structure of the set of rational numbers. They are guided to see that all the basic properties of operations with whole numbers appear to be maintained in the set of rationals. They find that the set of rational numbers is closed under division except for division by zero (whereas the set of whole numbers is not closed under division). Indeed this need for a set of numbers in which division (except division by 0) is always possible paves the way for the construction of the set of rational numbers.

# CHAPTER V

## RATIONAL NUMBERS (Multiplication and Division)

Lessons 45 - 60

### Lesson 45

Topic: Rational Numbers

Aim: To understand what is meant by a rational number

Specific Objectives:

To show the need for a new kind of number to make the division of any two whole numbers possible, except division by zero
Meaning of rational number
To develop the understanding that the set of whole numbers is a subset of the set of rational numbers
To develop the following relationships between fractions and rational numbers:
  a. A fraction is one of the symbols used to name a rational number
  b. Every rational number can be named by a fraction
  c. Many fractions name the same rational number

Challenge: Which of the following two problems has no solution if we are permitted to use only whole numbers?

Problem 1: Three boys earned \$6 mowing lawns. How much did each boy receive, if the money was shared equally?

Problem 2: Three boys earned \$5 mowing lawns. How much did each boy receive, if the money was shared equally?

I. Procedure

A. Need for a new kind of number

1. What is the solution to problem 1? (\$2) What kind of number is 2? (whole number)

2. What is the solution to problem 2? (there is no solution if we are permitted to use only whole numbers because 5÷3 is not a whole number.)

3. Have pupils suggest additional mathematical situations which could not be described if only whole numbers were used, as, for example,

   a. the amount of candy we get when we share a candy bar with a friend
   b. a homework assignment that takes less than 1 hour's time
   c. the length of a line segment that is more than 2 inches but less than 3 inches long.

4. Have pupils realize that if only whole numbers were used there would be no result when we perform computations such as $3 \div 4$ or $5 \div 3$. Elicit that a new kind of number is needed to express many mathematical situations.

B. Meaning of rational number

1. Tell pupils that the numbers we obtain when we divide a whole number by a counting number are called rational numbers.

2. Is $\frac{15}{2}$ a rational number? Why?

   Is $\frac{8}{4}$ a rational number? Why?

3. Elicit other examples of rational numbers.

C. Set of whole numbers is a subset of the set of rational numbers

1. Elicit that a whole number such as 5 can be expressed as:

   $\frac{5}{1}$ $\frac{10}{2}$ $\frac{15}{3}$ , and so on.

   Why may we consider every whole number a rational number? (Any whole number can be expressed as the quotient of a whole number by a counting number.)

2. What are the two possibilities when we divide a whole number by a counting number? Elicit that the result may be a whole number, as, for example, $\frac{8}{4}$ or 2; the result may be a fractional number, (other than a whole number), such as $\frac{15}{2}$ .

3. Guide pupils to see that every whole number is a rational number, although not every rational number is a whole number.

4. Have pupils see that every member of the set of whole numbers is a member of the set of rational numbers. Therefore, the set of whole numbers is a subset of the set of rational numbers.

D. Relationships between fractions and rational numbers

1. A fraction is one of the symbols used to express a rational number.

   a. Have pupils recall that symbols like "$\frac{1}{3}$", "$\frac{5}{3}$", "$\frac{2}{2}$" etc. are called fractions.

   b. Consider the rational number obtained when 3 is divided by 2. Elicit that this number may be expressed as:

      $\frac{3}{2}$, $1\frac{1}{2}$, 1.5

      Which of the above symbols is a fraction?

   c. After several such illustrations, elicit that a fraction is one of the symbols used to represent a rational number.

2. Every rational number can be named by a fraction.

   a. Consider the rational number $1\frac{2}{3}$. How else can we represent this number? ($\frac{5}{3}$)

   b. Consider the rational number .7. How else can we represent this number? ($\frac{7}{10}$)

   c. Have pupils conclude that because of the meaning of rational number every rational number can be named by a fraction.

   d. Have pupils realize that we may say that rational numbers are numbers which have fractions as names. However, not all rational numbers need be named by fractions.

3. Many fractions name the same rational number.

   a. Consider the rational number $\frac{2}{3}$. What other fractions can be used to name $\frac{2}{3}$? ( $\frac{4}{6}$, $\frac{6}{9}$, and so on )

   b. Name several fractions that symbolize zero.

   ( $\frac{0}{1}$, $\frac{0}{2}$, $\frac{0}{3}$, etc.)

   c. Is $2\frac{3}{5}$ the symbol for a rational number? Which one? List several other names for this rational number.

   d. Elicit that many fractions name the same rational number.

II. Practice

A. Write $3 \div 8$ in fractional form.

B. Write three names for 4, using fractions.

C. Write two other names for each of the following rational numbers:

1. $\frac{1}{2}$ (Answer: $\frac{2}{4}$, $\frac{4}{8}$, .5, etc.)
2. $\frac{2}{3}$
3. $\frac{3}{2}$
4. $\frac{4}{5}$
5. 3
6. $1\frac{1}{4}$

D. Which of the following do not represent rational numbers?

1. $\frac{2}{3}$
2. $\frac{0}{8}$
3. $\frac{111}{1076}$
4. $\frac{1}{0}$ (this does not because 0 is not a counting number)
5. 0
6. $3\frac{2}{5}$

E. Give three other names for the rational number named by $1\frac{2}{3}$.

F. Show that .5 names a rational number.

G. Mark each statement true (T) or False (F).

1. Every whole number is a rational number.
2. Every rational number is a whole number.
3. Decimal fractions represent rational numbers.
4. The set of counting numbers is a subset of the set of rational numbers.

H. Make a Venn diagram to show that the set of whole numbers is a subset of the set of rational numbers.

III. Summary

A. What is a rational number?

B. Why do we need rational numbers?

C. Why is the set of whole numbers a subset of the set of rational numbers?

D. How many fractions name each rational number?

E. What new vocabulary have we learned today? (rational number)

## Lesson 46

**Topic:** Rational Numbers

**Aim:** To learn to compute the product of two (or more) rational numbers

**Specific Objectives:**

To show the need for a method of multiplying rational numbers
Computing the product of two (or more) rational numbers
Properties of multiplication of rational numbers: closure; commutativity; associativity

**Challenge:** Show how each of the following can be expressed as repeated addition.

1. $2 \times 3$ 2. $4 \times \frac{1}{2}$ 3. $\frac{1}{2} \times \frac{1}{4}$

I. Procedure

A. The need for a method of multiplying rational numbers

1. In examples 1 and 2 above, we can interpret multiplication as adding equal addends:

$2 \times 3 = 3 + 3$
$4 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

This meaning applies only when the multiplier is a whole number. Therefore, this meaning cannot be applied to example 3: $\frac{1}{2} \times \frac{1}{4}$

2. Therefore, we need a meaning of multiplication that will apply to all cases of multiplication with rational numbers. (challenge)

B. Computing the product of rational numbers.

1. What is meant by $\frac{2}{3} \times \frac{4}{5}$ ?

a. Discuss the fact that we would like the associative property of multiplication to hold for the multiplication of rational numbers.

b. Consider $15 \times \left( \frac{2}{3} \times \frac{4}{5} \right)$.

If the associative property for multiplication is to hold, then

$$15 \times \left( \frac{2}{3} \times \frac{4}{5} \right) = \left(15 \times \frac{2}{3}\right) \times \frac{4}{5}$$

$$= 10 \times \frac{4}{5} \quad \text{(Why?)}$$

$$= 8 \quad \text{(Why?)}$$

Then $15 \times \left( \frac{2}{3} \times \frac{4}{5} \right) = 8$

In order for this to be true, $\frac{2}{3} \times \frac{4}{5}$ must be equal to $\frac{8}{15}$. That is,

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$

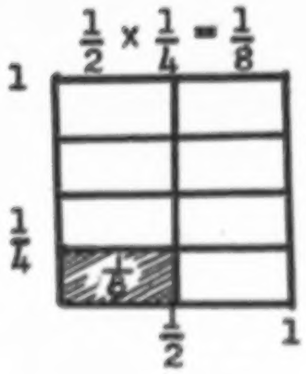
2. After several such illustrations have pupils see the justification in terms of basic properties for the rule for multiplying rational numbers as they did the fractional numbers of arithmetic.

3. If necessary, this procedure for computing the product of two rational numbers can be made to appear reasonable (that is, in agreement with the physical world) by thinking of $\frac{1}{2} \times \frac{1}{4}$ as $\frac{1}{4}$ divided into two equal parts and taking one of these equal parts.

Have pupils use diagrams to illustrate the following cases of multiplication of rational numbers.

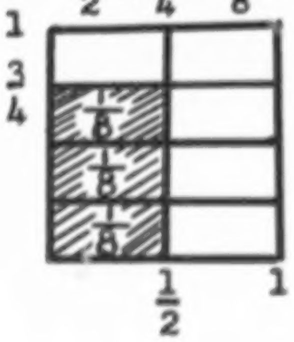
a.

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$



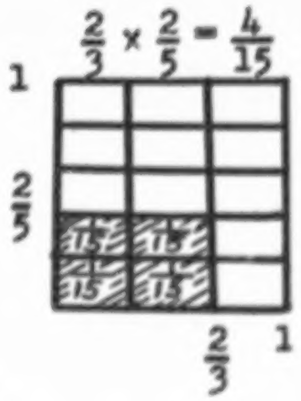
b.

$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$



c.

$\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$



4. Have pupils practice computing products of rational numbers.

   a. $\frac{2}{5} \times \frac{1}{2}$ c. $\frac{5}{3} \times \frac{7}{4}$

   b. $\frac{2}{3} \times \frac{2}{3}$ d. $4 \times \frac{2}{3}$ (4 can be expressed as $\frac{4}{1}$)

5. Extend the rule for computing the product of rational numbers to include more than two rational numbers.

$$\frac{1}{3} \times \frac{1}{5} \times \frac{2}{7} = \frac{1 \times 1 \times 2}{3 \times 5 \times 7} = \frac{2}{105}$$

C. Properties of Multiplication of Rational Numbers

1. Closure

   a. If we multiply rational numbers as we do fractional numbers, will the product always be a rational number? (Is the set of rational numbers closed under multiplication?)

      1) Have the pupils compute the products of the following rational numbers to see whether the products are rational numbers:

         $\frac{2}{3} \times \frac{1}{4}$ $\frac{5}{2} \times \frac{2}{3}$ $\frac{4}{1} \times \frac{3}{2}$ $\frac{3}{2} \times \frac{1}{3}$

         Is each of the products a rational number? Why?

      2) Test other cases by having the pupils suggest which rational numbers to multiply. Have pupils try to find a counter-example.

   b. Why can we assume that the product of two rational numbers will be a rational number? (We haven't been able to find a counter-example)

   c. (Optional) Have pupils see that the rule for computing the product of rational numbers insures that the product is a rational number.

      1) The product of the two numerators must be a whole number (Why?)

2) The product of the two denominators must be a counting number (Why?)

3) The product is a rational number (Why?)

2. Commutativity

a. Is multiplication of rational numbers commutative? For example, does $\frac{2}{3} \times \frac{2}{5} = \frac{2}{5} \times \frac{2}{3}$?

1) $\frac{2}{3} \times \frac{2}{5} = \frac{2 \times 2}{3 \times 5} = \frac{4}{15}$

2) $\frac{2}{5} \times \frac{2}{3} = \frac{2 \times 2}{5 \times 3} = \frac{4}{15}$

3) Therefore, $\frac{2}{3} \times \frac{2}{5} = \frac{2}{5} \times \frac{2}{3}$

b. After several such illustrations, have pupils conclude that multiplication of rational numbers is commutative.

c. (Optional) Try to establish the generalization from commutativity of multiplication of whole numbers.

Thus, $\frac{2}{3} \times \frac{3}{5} = \frac{2 \times 3}{3 \times 5} = \frac{3 \times 2}{5 \times 3} = \frac{3}{5} \times \frac{2}{3}$

3. Associativity

a. Is multiplication of rational numbers associative? For example,

does $(\frac{1}{2} \times \frac{1}{3}) \times \frac{1}{4} = \frac{1}{2} \times (\frac{1}{3} \times \frac{1}{4})$?

1) $(\frac{1}{2} \times \frac{1}{3}) \times \frac{1}{4} = \frac{1}{6} \times \frac{1}{4} = \frac{1}{24}$

2) $\frac{1}{2} \times (\frac{1}{3} \times \frac{1}{4}) = \frac{1}{2} \times \frac{1}{12} = \frac{1}{24}$

3) Therefore, $(\frac{1}{2} \times \frac{1}{3}) \times \frac{1}{4} = \frac{1}{2} \times (\frac{1}{3} \times \frac{1}{4})$

b. After several such illustrations, have pupils conclude that the multiplication of rational numbers is associative.

c. (Optional) Try to establish the generalization from associativity of multiplication of whole numbers.

II. Practice

A. Compute the products:

1. $\frac{2}{3} \times \frac{4}{7}$ 2. $\frac{3}{2} \times \frac{5}{4}$ 3. $5 \times \frac{2}{3}$ 4. $\frac{3}{4} \times \frac{5}{3}$

B. Compute the products:

1. $\frac{1}{4} \times \frac{3}{4} \times \frac{1}{5}$ 2. $\frac{1}{4} \times 5 \times \frac{1}{5}$ 3. $\frac{5}{3} \times \frac{3}{4} \times \frac{5}{2}$

C. Show that $(\frac{2}{3} \times \frac{1}{5}) \times \frac{3}{11} = \frac{2}{3} \times (\frac{1}{5} \times \frac{3}{11})$

D. What number would you use for the frame that will make each sentence true?

1. $\frac{1}{2} \times (\frac{3}{4} \times \frac{5}{7}) = (\square \times \frac{3}{4}) \times \frac{5}{7}$

2. $\square \times \frac{1}{2} = \frac{1}{2} \times 8$

E. A jar of milk contains $\frac{1}{4}$ of a quart. If $\frac{2}{3}$ of the milk in the jar is used, what fractional part of a quart is used?

F. Of all the students in a certain school $\frac{3}{5}$ study foreign languages; of this number $\frac{2}{9}$ study Italian. What fractional part of the students at this school study Italian?

III. Summary

A. How does the multiplication of rational numbers compare with the multiplication of the fractional numbers of arithmetic?

B. Illustrate the associative and commutative properties of multiplication with rational numbers.

C. What do we mean when we say the set of rational numbers is closed under multiplication?

Lessons 47 and 48

Topic: Rational Numbers

Aim: To express rational numbers in "simplest" form

Specific Objectives:

- The identity element for multiplication of rational numbers (multiplicative identity)
- Using the multiplicative identity to write different names for the same number
- The "simplest" name for a rational number
- How to express rational numbers in simplest form

Challenge: What is the simplest name for $\frac{79 \times 3}{79 \times 7}$ ?

I. Procedure

A. The identity element for multiplication of rational numbers

1. Have pupils recall that the number one is the identity element for multiplication of whole numbers.

   $1 \times 3 = 3 \times 1 = 3$

2. Is there an identity element for multiplication of rational numbers?

   a. Is $\frac{1}{1} \times \frac{2}{3} = \frac{2}{3} \times \frac{1}{1} = \frac{2}{3}$ a true statement?

   b. Is $1 \times \frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3}$ a true statement?

3. After several such illustrations, elicit that the identity element for multiplication of rational numbers is one.

B. Using the multiplicative identity to write different names for the same number

1. What are some names for the number one?

   $1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{\square} = \frac{\square}{8}$, and so on

2. Have pupils consider:

$$\frac{1}{2} = 1 \times \frac{1}{2}$$

$$= \frac{2}{2} \times \frac{1}{2}$$

$$= \frac{2 \times 1}{2 \times 2}$$

$$= \frac{2}{4}$$

a. How did we rename one?

b. Rename $\frac{1}{2}$ in three ways using $\frac{3}{3}$, $\frac{4}{4}$ and $\frac{5}{5}$ as names for one.

3. After several such illustrations, elicit that to find another name for some rational number, we multiply the rational number by one. We use whichever name for one we need:

$\frac{2}{2}$ or $\frac{3}{3}$ or $\frac{4}{4}$ or $\frac{5}{5}$, and so forth.

4. Rename each of the following in three ways using $\frac{3}{3}$, $\frac{5}{5}$ and $\frac{10}{10}$ as names for one.

a. $\frac{2}{5}$ b. $\frac{3}{4}$ c. $\frac{5}{8}$ d. $\frac{20}{27}$

5. Elicit that multiplying a rational number by one may change its name, but not the number itself.

C. The "simplest" name for a rational number

1. Consider $\frac{12}{16}$.

a. What are other names for this rational number?

Elicit that $\frac{6}{8}$, $\frac{3}{4}$, $\frac{24}{32}$, $\frac{36}{48}$, etc. are all names for the rational number $\frac{12}{16}$.

b. In which name do we find a numerator and denominator with no common factor other than 1? ( $\frac{3}{4}$ )

c. Tell pupils that the fraction in which the numerator and denominator are relatively prime is called the simplest name of the rational number.

2. Which of the following are the simplest names for rational numbers? Explain.

   a. $\frac{2}{3}$ b. $\frac{5}{8}$ c. $\frac{3}{12}$ d. $\frac{3}{4}$ e. $\frac{8}{10}$ f. $\frac{3}{9}$

3. What is the simplest name for $\frac{79 \times 3}{79 \times 7}$? (challenge)

$$\frac{79 \times 3}{79 \times 7} = \frac{79}{79} \times \frac{3}{7}$$

$$= 1 \times \frac{3}{7}$$

$$= \frac{3}{7}$$

$\frac{3}{7}$ is the simplest name for $\frac{79 \times 3}{79 \times 7}$. Why?

D. How to express rational numbers in simplest form

1. Consider $\frac{30}{48}$.

How can we use the multiplicative identity to express this rational number in simplest form?

a. $\frac{30}{48} = \frac{2 \times 15}{2 \times 24}$

$$= \frac{2}{2} \times \frac{15}{24}$$

$$= 1 \times \frac{15}{24}$$

$= \frac{15}{24}$ Is this in its simplest form? Explain.

b. $\frac{15}{24} = \frac{3 \times 5}{3 \times 8}$

$$= \frac{3}{3} \times \frac{5}{8}$$

$$= 1 \times \frac{5}{8}$$

$= \frac{5}{8}$ Is this in its simplest form? Explain.

c. Therefore, the simplest form of $\frac{30}{48}$ is $\frac{5}{8}$.

2. Have pupils practice expressing rational numbers in simplest form.

3. Pose question: Is there a shorter method of finding the simplest name for a rational number, as, for example, the rational number $\frac{30}{48}$ ?

   a. What is the greatest common factor of 30 and 48?

      1) $30 = 2 \times 3 \times 5$

      2) $48 = 2 \times 2 \times 2 \times 2 \times 3$

      3) The greatest common factor of 30 and 48 is $2 \times 3$, or 6

   b. Using the greatest common factor, rename 30 as $5 \times 6$ and 48 as $12 \times 6$. Then $\frac{30}{48}$ can be renamed as follows:

$$\frac{30}{48} = \frac{5 \times 6}{8 \times 6}$$

$$= \frac{5}{8} \times \frac{6}{6}$$

$$= \frac{5}{8} \times 1$$

$$= \frac{5}{8}$$

II. Practice

A. Write three other names for $\frac{12}{18}$. (e.g. $\frac{6}{9}$, $\frac{24}{36}$, $\frac{2}{3}$). What is the simplest name for $\frac{12}{18}$ ?

B. Are $\frac{15}{24}$ and $\frac{42}{54}$ names for the same rational number? Explain.

C. In each row, encircle the numeral that is not a name for the same rational number.

   1. $\frac{1}{2}$, $\frac{16}{32}$, $\frac{32}{66}$, $\frac{81}{162}$

2. $\frac{5}{8}, \frac{10}{16}, \frac{16}{24}, \frac{20}{32}$

3. $.75, \frac{6}{8}, \frac{75}{100}, \frac{8}{12}$

D. Find the greatest common factor of the numerator and denominator in each case and then find the simplest name for each rational number.

1. $\frac{16}{48}$ 2. $\frac{24}{36}$ 3. $\frac{64}{96}$ 4. $\frac{81}{117}$ 5. $\frac{74}{96}$

E. Are the numerators and denominators of each of the following relatively prime? Explain.

1. $\frac{4}{7}$ 2. $\frac{6}{9}$ 3. $\frac{27}{35}$ 4. $\frac{42}{56}$ 5. $\frac{28}{140}$

F. Find the simplest form of those rational numbers in E which are not already in simplest form.

III. Summary

A. What is the identity element for multiplication of rational numbers?

B. The number one has many names. How does this help us to write other names for a rational number?

C. Two boys measured a line segment. One said the measure of the segment in inches is $\frac{3}{4}$. The other said it is $\frac{12}{16}$. Are both correct? Why?

D. What is meant by the simplest name for a rational number?

E. How do we use the greatest common factor of the numerator and denominator of a rational number to express the number in simplest form?

F. What new vocabulary have you learned today? (simplest name of a rational number, multiplicative identity)

## Lessons 49 and 50

Topic: Multiples of Numbers

Aim: To develop the concept of multiple, common multiple, and least common multiple

Specific Objectives:

Meaning of multiple
Meaning of common multiple
Obtaining common multiples
Obtaining the least common multiple by inspection; by prime factorization

Note to Teacher: By the multiplicative property of zero, ($a \times 0 = 0 \times a = 0$ for all numbers), zero is a multiple of every whole number. Zero as a common multiple is of limited value. We will use as the least common multiple of two (or more) numbers, the smallest non-zero whole number which is a common multiple of the numbers considered.

Challenge: Some multiples of 10 are: 10, 20, 30, 40, 50
Name the next three multiples of 10 in order.

I. Procedure

A. Meaning of multiple

1. Multiply 2 by: 1, 2, 3, 4, 5, 6, 12, 13, 15

   Tell pupils that the products 2, 4, 6, 8, 10, 12, 24, 26, 30 are called multiples of 2.

2. Multiply 3 by: 1, 2, 3, 4, 5, 7, 9, 12

   Tell pupils that the products 3, 6, 9, 12, 15, 21, 27, 36 are called multiples of 3.

3. Multiply 5 by: 1, 2, 3, 4, 6, 10, 14

   Tell pupils that the products 5, 10, 15, 20, 30, 50, 70 are called multiples of 5.

4. What factor do the multiples of 2 have in common? The multiples of 3? The multiples of 5?

   Elicit that a multiple of a number has that number as a factor. We may also say that a multiple of a number is divisible by the number.

5. Return to challenge: Why are 60, 70, 80 multiples of 10?

6. List ten multiples of 4; of 7; of 9.

B. Meaning of common multiple

1. List the multiples of 2 from 1 to 20.

   2, 4, 6, 8, 10, 12, 14, 16, 18, 20

2. List the multiples of 3 from 1 to 20.

   3, 6, 9, 12, 15, 18

3. What multiples in the above lists do the numbers 2 and 3 have in common? (6, 12, 18)

   Is 6 divisible by 2? by 3?
   Is 12 divisible by 2? by 3?
   Is 18 divisible by 2? by 3?

4. What is the set of all the multiples of 2?

   $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, \ldots\}$

5. What is the set of all the multiples of 3?

   $B = \{3, 6, 9, 12, 15, 18 \ldots\}$

6. What is the intersection set of sets A and B?

   $A \cap B = \{6, 12, 18, \ldots\}$

7. Elicit that the set $A \cap B$ is the set of all common multiples of 2 and 3.

C. Obtaining common multiples

1. How can we find the common multiples of two numbers, such as 3 and 5?

   Elicit that we can list the members of the set of multiples of 3, and the members of the set of multiples of 5. We would then determine the set of common multiples by inspection. (See B 4-7)

2. How can we obtain a common multiple of 3 and 5 without listing multiples of each number and inspecting them?

Elicit that the product of 3 and 5, that is, 15, is a common multiple of 3 and 5. Why? (15 has both 3 and 5 as factors and is, therefore, a multiple of each)

3. List the members of the set of common multiples less than 100 for each of the following pairs of numbers.

   a. 10, 15    b. 12, 16    c. 10, 11    d. 16, 20

D. Finding the least common multiple

1. How can we find the least common multiple of two numbers, such as 6 and 9?

   a. What is the set of all multiples of 6?

      A = {6, 12, 18, 24, 30, 36, ...}

   b. What is the set of all multiples of 9?

      B = {9, 18, 27, 36, 45, ...}

   c. What is the intersection of these sets?

      A ∩ B = {18, 36, ... }

   d. What is the least number in the intersection set? (18)

      Therefore, 18 is the least common multiple of 6 and 9. We have found it by inspection.

2. Use similar procedures to have pupils find the least common multiple of 3 and 7.

   a. The intersection set of multiples of 3 and 7 is {21, 42, ...}.

   b. What is the least number in the intersection set? (21)

      Therefore 21 is the least common multiple of 3 and 7.

3. Elicit that we can find the least common multiple of two or more numbers by finding the least number in the intersection set.

4. We know how to find the least common multiple of 6 and 9 by inspection. Is there another method?

a. Factor 6 completely: $6 = 2 \times 3$
   Factor 9 completely: $9 = 3 \times 3$

b. Elicit that the least common multiple of 6 and 9 (or any multiple of 6 and 9) must be divisible by 6, or $2 \times 3$, and by 9, or $3 \times 3$.

c. To be divisible by $2 \times 3$, a number must have the factors 2 and 3. To be divisible by $3 \times 3$, a number must have the factors 3 and 3. The number $2 \times 3 \times 3$, or 18, has such factors. (Since 3 already appears as a factor of 6, we need include it only once more as a factor of 9.)

d. Is 18 a multiple of 6? of 9? Elicit that 18 is a common multiple of 6 and 9.

e. Is 18 the least common multiple of 6 and 9? Why? (18 is the least counting number divisible by both 6 and 9).

5. What is the least common multiple of 12, 14, and 15?

   a. Factor 12 completely: $12 = 2^2 \times 3$
      Factor 14 completely: $14 = 2 \times 7$
      Factor 15 completely: $15 = 3 \times 5$

   b. Which factors must the least common multiple of 12, 14, and 15 contain?

      1) To be divisible by $2^2 \times 3$, the least common multiple must have the factors $2^2$ and 3.

      2) To be divisible by $2 \times 7$, the least common multiple must have the factors 2 and 7. The number $2^2 \times 3 \times 7$ has these factors. (Why do we not include 2 as a factor again?)

      3) To be divisible by $3 \times 5$, the least common multiple must have the factors 3 and 5. The number $2^2 \times 3 \times 5 \times 7$ has these factors. (Why do we not include 3 as a factor twice?)

   c. Thus, $2^2 \times 3 \times 5 \times 7$, or 420, is the least number divisible by 12, by 14, and by 15.

   d. Have pupils check to see that this is so.

## II. Practice

A. List the members of the set of multiples less than 50 for each of the following numbers.

1. 4     2. 6     3. 8     4. 9

B. List the members of the set of common multiples less than 100 for each of the following pairs of numbers.

1. 4, 6     2. 6, 9     3. 9, 16     4. 10, 11

C. For each pair of numbers in B, find the least common multiple.

D. For each of the following sets of numbers, find the least common multiple of the numbers in that set.

1. {4, 12}     2. {25, 35}     3. {8, 12, 15}     4. {3, 11, 19}

III. Summary

A. How do we obtain a multiple of a number?

B. What is meant by a common multiple of two numbers?

C. What is meant by the least common multiple of two (or more) numbers?

D. How can we obtain the least common multiple of two (or more) numbers?

E. How do you find the least common multiple of two different primes? (Find their product.)

F. Why is the least common multiple of two even numbers not their product? (2 is a common factor of the numbers.)

G. What new vocabulary have you learned today? (multiple, common multiple, least common multiple)

## Lessons 51 and 52

Topic: Rational Numbers

Aim: To compare two rational numbers

Specific Objectives:

To review the meaning of the inequality symbols: " > " and " < ".
To review how fractional numbers with like denominators can be compared.
To compare rational numbers with unlike denominators.

Challenge: Which is greater, $\frac{2}{3}$ or $\frac{5}{8}$ ?

I. Procedure

A. Inequality symbols " > " and " < "

1. Review meaning of the following symbols of inequality:

> "is greater than"
< "is less than"

2. Write in symbols:

Seventeen is greater than twelve. (17 > 12)
Five is less than eight. (5 < 8)
Eight-tenths is greater than two-thirds.
One and one-half is less than two.

3. What is the meaning of each of the following?

a. $9 > 6$ b. $10 > 2\frac{1}{2}$ c. $.3 < .7$ d. $1.4 > 1$

4. T[illegible] of the following, place a symbol, " > " or [illegible], between each pair of numerals so that a true statement results.

a. 8 __ 1 b. 4 __ 7 c. $3\frac{1}{2}$ __ 5 d. 2.5 __ 1.3

B. Comparing fractional numbers with like denominators

1. Which of the following fractions represents the rational number of least value?

$\frac{1}{8}$ $\frac{3}{8}$ $\frac{5}{8}$ $\frac{7}{8}$

a. Have pupils recall that the denominator of a fraction tells us the number of parts of equal size into which a unit or set has been divided.

The numerator of a fraction indicates the number of these parts with which we are concerned.

b. Why can $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ be compared? (Each indicates a number of eighths.)

c. Why is $\frac{1}{8}$ the fraction that represents the least value? (It stands for the smallest number of eighths.)

2. Elicit that if two fractional numbers have the same denominator, the fractional number with the lesser numerator is less than the fractional number with the greater numerator.

3. Which of these statements are true? Explain.

a. $\frac{2}{3} < \frac{7}{3}$ b. $\frac{4}{5} < \frac{2}{5}$ c. $\frac{1}{4} < 1\frac{1}{4}$ d. $\frac{5}{7} < \frac{2}{7}$

C. Comparing rational numbers with unlike denominators

1. Refer to challenge. Which is greater, $\frac{2}{3}$ or $\frac{5}{8}$ ?

Elicit that is is difficult to compare the values represented by these fractions since the denominators are unlike. If, however, we can rename $\frac{2}{3}$ and $\frac{5}{8}$ and express them as fractions with like denominators, we would then have only to compare the numerators.

2. How can we rename $\frac{2}{3}$ and $\frac{5}{8}$ as fractions with like denominators?

a. Have pupils use the multiplicative identity and write names for $\frac{2}{3}$ and $\frac{5}{8}$ until they find names with a common denominator.

$$\frac{2}{3} \times 1 = \frac{2}{3}$$

$$\frac{2}{3} \times \frac{2}{2} = \frac{2\times2}{3\times2} = \frac{4}{6}$$

$$\frac{2}{3} \times \frac{3}{3} = \frac{2\times3}{3\times3} = \frac{6}{9}$$

$$\frac{2}{3} \times \frac{4}{4} = \frac{2\times4}{3\times4} = \frac{8}{12}$$

$$\frac{2}{3} \times \frac{5}{5} = \frac{2\times5}{3\times5} = \frac{10}{15}$$

$$\frac{2}{3} \times \frac{6}{6} = \frac{2\times6}{3\times6} = \frac{12}{18}$$

$$\frac{2}{3} \times \frac{7}{7} = \frac{2\times7}{3\times7} = \frac{14}{21}$$

$$\frac{2}{3} \times \frac{8}{8} = \frac{2\times8}{3\times8} = \frac{16}{24}$$

$$\frac{5}{8} \times 1 = \frac{5}{8}$$

$$\frac{5}{8} \times \frac{2}{2} = \frac{10}{16}$$

$$\frac{5}{8} \times \frac{3}{3} = \frac{15}{24}$$

b. Elicit that $\frac{16}{24}$ is another name for $\frac{2}{3}$, and $\frac{15}{24}$ another name for $\frac{5}{8}$. Why can the fractions now be compared? ( $\frac{15}{24}$ and $\frac{16}{24}$ have a common denominator.)

Since $\frac{16}{24} > \frac{15}{24}$, we can now say that $\frac{2}{3} > \frac{5}{8}$.

3. What is another way of renaming $\frac{2}{3}$ and $\frac{5}{8}$ so that they have a common denominator?

   a. What are the denominators of $\frac{2}{3}$ and $\frac{5}{8}$? (3 and 8)

   b. Elicit that a common denominator must be a multiple of 3 and of 8.

      1) What is the least common multiple of the denominators of $\frac{2}{3}$ and $\frac{5}{8}$? (24)

      2) Express $\frac{2}{3}$ and $\frac{5}{8}$ as fractions with the common denominator, 24.

$$\frac{2}{3} \times 1 = \frac{2}{3} \qquad\qquad \frac{5}{8} \times 1 = \frac{5}{8}$$

$$\frac{2}{3} \times \frac{8}{8} = \frac{2 \times 8}{3 \times 8} \qquad\qquad \frac{5}{8} \times \frac{3}{3} = \frac{5 \times 3}{8 \times 3}$$

$$\frac{2}{3} = \frac{16}{24} \qquad\qquad \frac{5}{8} = \frac{15}{24}$$

   c. Elicit that since the fractions now have a common denominator, it is possible to compare them. Since $\frac{16}{24} > \frac{15}{24}$, we can say that $\frac{2}{3} > \frac{5}{8}$.

4. Have pupils generalize: If two fractions have unlike denominators, the fractional numbers they represent can be compared by expressing them as fractions with a common denominator.

II. Practice

A. Circle the fraction in each pair that represents the greater value:

1. $\frac{4}{7}, \frac{6}{7}$

3. $\frac{9}{32}, \frac{17}{32}$

2. $\frac{8}{15}, \frac{6}{15}$

4. $\frac{8}{17}, \frac{11}{17}$

B. Find a common denominator for each of the following pairs of fractions:

1. $\frac{1}{3}$ and $\frac{1}{5}$

3. $\frac{1}{2}$ and $\frac{5}{9}$

2. $\frac{1}{4}$ and $\frac{1}{8}$

4. $\frac{3}{7}$ and $\frac{2}{3}$

C. Which of these statements are true?

1. $\frac{2}{4} < \frac{4}{7}$

3. $\frac{11}{15} < \frac{3}{4}$

2. $\frac{7}{9} > \frac{3}{4}$

4. $\frac{5}{6} > \frac{4}{5}$

D. Use the symbols " > ", " < ", " = ", to complete each of the following so that a true statement results.

1. $\frac{3}{4}$ ___ $\frac{7}{9}$ $(\frac{3}{4} < \frac{7}{9})$

4. $\frac{7}{12}$ ___ $\frac{17}{24}$

2. $\frac{11}{15}$ ___ $\frac{2}{3}$

5. $\frac{2}{5}$ ___ $\frac{2}{6}$

3. $\frac{3}{8}$ ___ $\frac{36}{36}$

6. $\frac{4}{3}$ ___ $\frac{8}{6}$

E. Which of the following fractions represents the greatest value?

1. $\frac{13}{24}$ 2. $\frac{8}{12}$ 3. $\frac{3}{4}$

F. Which of the following fractions names the least number?

1. $\frac{7}{8}$ 2. $\frac{4}{5}$ 3. $\frac{36}{40}$

4

G. Which of these statements are true?

1. $\frac{1}{3} < \frac{1}{2}$ 3. $\frac{2}{7} < \frac{2}{5}$ 5. $\frac{5}{9} < \frac{5}{7}$

2. $\frac{1}{8} < \frac{1}{6}$ 4. $\frac{3}{10} < \frac{3}{5}$ 6. $\frac{11}{8} < \frac{11}{6}$

H. Study the numerators in the statements in G. From the pattern you observe, tell how you can compare rational numbers which have the same numerator.

III. Summary

A. What symbols of inequality have we reviewed today?

B. How can rational numbers with like denominators be compared?

C. How can rational numbers with unlike denominators be compared?

D. Explain how you would express rational numbers with unlike denominators as fractions with the same denominator.

## Lessons 53 and 54

**Topic:** Rational Numbers and the Number Line

**Aim:** To develop an understanding of how rational numbers are associated with points on the number line

**Specific Objectives:**

Visualizing the rational numbers as points on a line
Order and the number line
The Density Property of the set of rational numbers

**Challenge:** How many rational numbers are there between 0 and 1?

I. Procedure

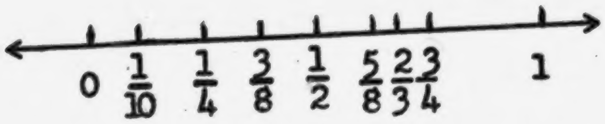
A. The rational numbers on the number line

1. Have pupils recall that the whole numbers are associated with some points on the number line.

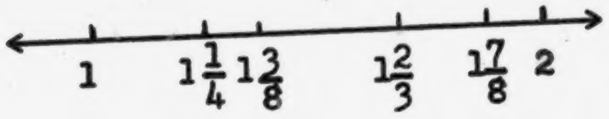
2. Have them see that between any two points associated with two consecutive whole numbers, say 0 and 1, there are many more points on the number line.

   a. What number would you use to mark a point half-way between 0 and 1? Two-thirds of the way between 0 and 1?

   b. Locate points on the number line which are associated with rational numbers between 0 and 1 as in the following diagram.

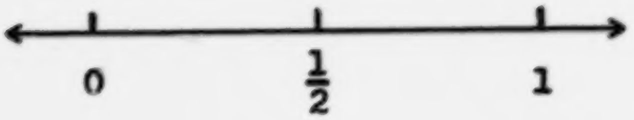


3. Have pupils repeat this procedure with other intervals on the number line, say, the interval between 1 and 2, 2 and 3, etc.



4. Have pupils realize that certain points on the number line are associated with rational numbers.

5. Have them consider the point on the number line associated with the rational number $\frac{1}{2}$. Where is it located?



   a. Where is the location of the point associated with $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots$?

   b. Elicit that since the numerals $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$ are names for the number $\frac{1}{2}$, each is associated with the same point on the number line.

6. Where is the location of the point associated with $1, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots$? With what point on the number line is each of these associated?

B. Order and the number line

   1. Elicit that given two different rational numbers the first is greater than the second or less than the second. This is called the property of order of the set of rational numbers.

   2. Arrange the numbers $\frac{1}{3}, \frac{3}{5}, \frac{2}{4}, \frac{3}{10}, \frac{7}{8}, \frac{4}{9}$ in order from least to greatest.

   3. How do we show the ordering of the set of rational numbers when we assign rational numbers to points on the number line?

      a. Have pupils consider the points on the number line associated with the rational numbers $1\frac{3}{4}$ and $1\frac{1}{2}$.

b. Which is the greater of the two numbers? ($1\frac{3}{4} > 1\frac{1}{2}$)

c. How is the point associated with $1\frac{3}{4}$ located with respect to the point associated with $1\frac{1}{2}$? (The point associated with $1\frac{3}{4}$ is to the right of the point associated with $1\frac{1}{2}$.)

4. After several such illustrations, elicit that given two different rational numbers, the greater number is associated with a point which is to the right of the point associated with the lesser number.

C. Density of the set of rational numbers

1. Is there a whole number greater than 0 and less than 2? Name it. (1) (We say that 1 is between 0 and 2.)

2. Is there a whole number greater than 0 and less than 1? (There is no whole number between 0 and 1.)

   Elicit that between any two whole numbers there is not always another whole number.

3. Is there a rational number greater than 0 and less than 1? Name several such numbers.

   ($\frac{1}{2}, \frac{1}{5}, \frac{1}{4}, \frac{2}{3}, \frac{5}{6}, \frac{9}{11}$, and so on)

4. Consider the rational number $\frac{1}{2}$ . Elicit that $\frac{1}{2}$ is between 0 and 1 because $0 < \frac{1}{2}$ and $\frac{1}{2} < 1$.

   a. Is there a rational number between $\frac{1}{2}$ and 1? One such number is the number half-way between $\frac{1}{2}$ and 1. It can be found by taking the average of the two numbers.

   $(\frac{1}{2} + 1) \div 2 = \frac{3}{4}$ $\frac{3}{4}$ is a rational number between $\frac{1}{2}$ and 1.

   b. Is there a rational number between $\frac{3}{4}$ and 1? One such number is the number half-way between $\frac{3}{4}$ and 1.

$(\frac{3}{4} + 1) \div 2 = \frac{7}{8}$ $\frac{7}{8}$ is a rational number between $\frac{3}{4}$ and 1.

5. After several such illustrations, have pupils realize that between any two rational numbers there is another rational number.

6. Refer to challenge. How many rational numbers are there between 0 and 1? Elicit that since between any two rational numbers there is another rational number, then between any two rational numbers there exists an infinite number of rational numbers.

7. Tell pupils that this property of rational numbers is called density. We say that the set of rational numbers is a dense set, because between any two rational numbers there are infinitely many such rational numbers.

   Note: Although we associate every rational number with a point on the number line, we have not accounted for every point on the number line, as pupils will learn in a future grade.

8. Is the set of whole numbers a dense set? Why not?

I. Practice

A. Draw a picture of a part of a number line between 0 and 2 and mark on it the points corresponding to the following numbers.

1. $\frac{2}{3}$
2. $\frac{5}{8}$
3. $1\frac{1}{2}$
4. $\frac{31}{16}$
5. $1\frac{3}{5}$
6. $1\frac{7}{8}$

B. Refer to the number line in A and tell which number in each pair is the greater.

1. $\frac{5}{8}$ and $\frac{2}{3}$ 2. $1\frac{1}{2}$ and $1\frac{3}{5}$ 3. $\frac{31}{16}$ and $1\frac{7}{8}$

C. Give a rational number between

1. 1 and 2 Solution: $1\frac{1}{2}$ or 1.3
2. 3 and 4
3. $1\frac{1}{2}$ and $1\frac{3}{4}$

4. $2\frac{3}{8}$ and $3\frac{1}{4}$

5. $\frac{5}{8}$ and $\frac{2}{3}$

### III. Summary

A. What can you say about a point associated with:

$3, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \ldots$ ? Explain.

B. Given two different rational numbers of which the first is greater than the second. What can you say about the point on the number line corresponding to the second number? (It is to the left of the point for the first number.)

C. If we agree that with every rational number we can associate a point on the number line, how many points are there between 0 and 1 on the number line?

D. What new vocabulary have you learned today? (order, density)

## Lesson 55

**Topic:** Multiplication of Rational Numbers

**Aim:** To multiply rational numbers expressed in mixed fractional form (mixed numerals).

**Specific Objectives:**

To review the multiplicative property of 1.
To review renaming a number expressed in mixed fractional form, as an improper fraction.
To multiply rational numbers expressed in mixed fractional form.

Challenge: A candy recipe calls for $2\frac{2}{3}$ cups of sugar. How much sugar is needed to make twice as much candy?

I. Procedure

A. Review the multiplicative property of one.

1. What is the product of 1 and any rational number? (the number)

2. What are some names for the identity element of multiplication?

   $1, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}$, and so on

B. Renaming a number expressed in mixed fractional form, as an improper fraction.

1. Have pupils recall that a number such as $2\frac{2}{3}$ can be expressed as $2 + \frac{2}{3}$.

2. $2\frac{2}{3} = 2 + \frac{2}{3}$

   $= (2 \times 1) + \frac{2}{3}$

   $= (2 \times \frac{3}{3}) + \frac{2}{3}$ Why was $\frac{3}{3}$ chosen as another name for 1?

   $= \frac{6}{3} + \frac{2}{3}$

   $= \frac{8}{3}$

   Thus $\frac{8}{3}$ is another name for $2\frac{2}{3}$.

3. Have pupils practice renaming numbers expressed in mixed fractional form, as improper fractions.

a. In each of the following, replace the frames.

$$\begin{aligned}3\tfrac{2}{3} &= 3 + \tfrac{2}{3}\\ &= (3 \times 1) + \tfrac{2}{3}\\ &= (3 \times \tfrac{\square}{\square}) + \tfrac{2}{3}\\ &= \tfrac{9}{3} + \tfrac{2}{3}\\ &= \tfrac{11}{3}\end{aligned}$$

$$\begin{aligned}4\tfrac{1}{8} &= 4 + \tfrac{1}{8}\\ &= (4 \times 1) + \tfrac{1}{8}\\ &= (4 \times \tfrac{\square}{\square}) + \tfrac{1}{8}\\ &= \tfrac{32}{8} + \tfrac{1}{8}\\ &= \tfrac{33}{8}\end{aligned}$$

$$\begin{aligned}2\tfrac{5}{6} &= 2 + \tfrac{5}{6}\\ &= (2 \times 1) + \tfrac{5}{6}\\ &= (2 \times \tfrac{\square}{\square}) + \tfrac{5}{6}\\ &= \Diamond + \tfrac{5}{6}\\ &= \tfrac{17}{6}\end{aligned}$$

In each case, explain why the particular form of 1 was used.

b. Express the following rational numbers as improper fractions:

1) $2\frac{5}{9}$ 2) $3\frac{3}{4}$ 3) $5\frac{1}{8}$ 4) $1\frac{15}{16}$

c. Express the following rational numbers as mixed numerals:

1) $\frac{14}{3}$ 2) $\frac{20}{6}$ 3) $\frac{12}{5}$ 4) $\frac{17}{8}$

C. Multiplication of rational numbers expressed in mixed fractional form

1. Refer to challenge.

To solve the problem it is necessary to compute the following:

$$\begin{aligned}2 \times 2\tfrac{2}{3} &= 2 \times \tfrac{8}{3}\\ &= \tfrac{2}{1} \times \tfrac{8}{3}\\ &= \tfrac{2 \times 8}{1 \times 3} = \tfrac{16}{3}\end{aligned}$$

$\frac{16}{3}$ can be expressed as $5\frac{1}{3}$.

We will need $5\frac{1}{3}$ cups of sugar.

2. Consider: $3\frac{3}{4} \times 2\frac{1}{2}$.

Have pupils perform the computation as follows:

$$3\frac{3}{4} \times 2\frac{1}{2} = \frac{15}{4} \times \frac{5}{2}$$

$$= \frac{75}{8}$$

$$= 9\frac{3}{8}$$

3. Consider: $4\frac{2}{3} \times 1\frac{1}{4}$

Have pupils perform the computation as follows:

$$4\frac{2}{3} \times 1\frac{1}{4} = \frac{14}{3} \times \frac{5}{4}$$

$$= \frac{14 \times 5}{3 \times 4}$$

$$= \frac{70}{12}$$

$= 5\frac{10}{12}$ Is $\frac{10}{12}$ in simplest form?

$$= 5\frac{5}{6}$$

4. After several similar examples, elicit that to multiply rational numbers expressed as mixed numerals, we first rename the number as an improper fraction, and then we multiply.

II. Practice

A. Replace the frames to make the resulting statements true:

1. $4\frac{3}{7} = \frac{\square}{7}$
2. $3 = \frac{\square}{6}$
3. $\frac{9}{3} = \square$
4. $2\frac{5}{8} = \square + \frac{5}{8}$
5. $3 = \frac{\square}{4}$
6. $\frac{13}{4} = \square + \frac{1}{4}$
7. $\frac{14}{6} = 2 + \square$
8. $\frac{27}{8} = 3\frac{\square}{2}$

B. Rename the following as mixed fractions:

1. $\frac{11}{4}$  2. $\frac{13}{5}$  3. $\frac{7}{3}$  4. $\frac{39}{4}$

C. Multiply as indicated:

1. $5 \times 6\frac{1}{6}$  2. $3\frac{1}{8} \times 2\frac{3}{4}$  3. $1\frac{7}{8} \times 3\frac{2}{3}$  4. $2\frac{1}{2} \times 3\frac{1}{4}$

D. Mary brought home $2\frac{1}{2}$ dozen seashells after a visit to the shore. If she gave $\frac{1}{3}$ of her shells to the class shell collection, how many dozen shells did she contribute?

E. One and one-quarter feet of wood are needed to make a pair of book-ends. How many feet of wood are needed so that each of 5 boys can make a pair of book-ends?

F. Consult suitable textbooks for additional practice material.

III. Summary

A. How many different names are there for the multiplicative identity 1? (an infinite number, e.g., $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$ )

B. How can you decide which of these names for the multiplicative identity to use when you are renaming a rational number expressed as a mixed numeral?

C. What procedure would you use in multiplying rational numbers expressed as mixed numerals?

## Lessons 56 and 57

Topic: Multiplication of Rational Numbers

Aim: To use the multiplicative identity to simplify finding the product of two (or more) rational numbers

Specific Objectives:

To use the multiplicative identity to shorten our work in multiplying rational numbers
To use the concept of greatest common factor to simplify computation in the multiplication of rationals

Challenge: A boy walks at the rate of $2\frac{1}{3}$ miles per hours. At this rate, how far will he walk in $1\frac{1}{2}$ hours?

I. Procedure

A. Using the multiplicative identity

1. Refer to challenge. Guide pupils to see that to solve the problem we perform the following computation.

a. $1\frac{1}{2} \times 2\frac{1}{3} = \frac{3}{2} \times \frac{7}{3}$ ($\frac{3}{2}$ is another name for $1\frac{1}{2}$, etc.)

$= \frac{3 \times 7}{2 \times 3}$ (Multiplication of rational numbers)

$= \frac{21}{6}$ Is this in simplest form? Explain.

b. Since $\frac{21}{6}$ is not in simplest form, we continue as follows:

$\frac{21}{6} = \frac{3 \times 7}{2 \times 3}$ (Factoring)

$= \frac{3 \times 7}{3 \times 2}$ (Commutative property of multiplication)

$= \frac{3}{3} \times \frac{7}{2}$ (Multiplication of rational numbers)

$= 1 \times \frac{7}{2}$ ($\frac{3}{3}$ is another name for 1)

$= \frac{7}{2}$ or $3\frac{1}{2}$ (Multiplicative identity)

c. Elicit that the computation may be simplified if we use the multiplicative identity before we compute the products of numerators and denominators. We would accordingly proceed as follows:

$1\frac{1}{2} \times 2\frac{1}{3} = \frac{3}{2} \times \frac{7}{3}$ ($\frac{3}{2}$ is another name for $1\frac{1}{2}$, etc.)

$= \frac{3 \times 7}{2 \times 3}$ (Multiplication of rational numbers)

$= \frac{3 \times 7}{3 \times 2}$ (Commutative property of multiplication)

$= \frac{3}{3} \times \frac{7}{2}$ (Multiplication of rational numbers)

$= 1 \times \frac{7}{2}$ ($\frac{3}{3}$ is another name for 1)

$= \frac{7}{2}$, or $3\frac{1}{2}$ (Multiplicative identity)

Is $\frac{7}{2}$ the simplest form of the product? Explain.

Note: Pupils should be able to justify each step, but need not record the justification as they compute. After pupils develop understanding, a shortened form of computation may be used.

2. Have pupils compute and express in simplest form several products such as the following:

a. $\frac{2}{3} \times \frac{3}{5}$

b. $\frac{6}{7} \times \frac{5}{6}$

c. $4 \times 1\frac{3}{4}$

d. $5\frac{1}{3} \times 2\frac{3}{16}$

3. Sometimes the use of the multiplicative identity in finding the simplest form of a product is not obvious. Let's see how we can use the multiplicative identity in the following examples: (Justify each step by giving a reason.)

a. Compute the following product and express in simplest form.

$\frac{5}{6} \times \frac{3}{8} = \frac{5 \times 3}{6 \times 8}$ (Definition of multiplication of rational numbers)

$= \frac{3 \times 5}{6 \times 8}$ ( Commutative property of multiplication)

$= \frac{3 \times 5}{(3 \times 2) \times 8}$ (3 and 2 are factors of 6)

$= \frac{3 \times 5}{3 \times (2 \times 8)}$ (Associative property of multiplication)

$= \frac{3}{3} \times \frac{5}{2 \times 8}$ (Definition of multiplication of rational numbers)

$= 1 \times \frac{5}{16}$ ( $\frac{3}{3}$ is another name for 1)

$= \frac{5}{16}$ (1 is the multiplicative identity)

Is $\frac{5}{16}$ the simplest form of the product? Explain.

b. Why were we helped in this computation by writing 6 as 3 x 2? (We were able to see the multiplicative identity more easily.)

4. Have pupils compute and express in simplest form several products such as the following:

a. $\frac{3}{4} \times \frac{8}{11}$

c. $3 \times 3\frac{5}{6}$

b. $\frac{2}{3} \times \frac{3}{4}$

d. $3\frac{1}{3} \times 1\frac{3}{4}$

B. Using the concept of greatest common factor to find the simplest form of a product

1. Consider $\frac{3}{2} \times \frac{14}{15}$. What is the simplest form of the product?

$\frac{3}{2} \times \frac{14}{15} = \frac{3 \times 14}{2 \times 15}$ What shall we do to be able to find more easily the multiplicative identity, if there is one? (Factor into primes)

$= \frac{3 \times 2 \times 7}{2 \times 3 \times 5}$ Elicit that 2 x 3 is the greatest common factor of numerator and denominator.

$= \frac{2 \times 3}{2 \times 3} \times \frac{7}{5}$

$= 1 \times \frac{7}{5}$

$= \frac{7}{5}$ or $1\frac{2}{5}$ Is $\frac{7}{5}$ the simplest form of the product?

Explain.

2. Consider $\frac{3}{10} \times \frac{6}{9}$. What is the simplest form of the product?

$$\frac{3}{10} \times \frac{6}{9} = \frac{3 \times 6}{10 \times 9}$$

$$= \frac{3 \times 2 \times 3}{2 \times 5 \times 3 \times 3}$$

What is the greatest common factor of numerator and denominator?

$$= \frac{2 \times 3 \times 3}{2 \times 3 \times 3} \times \frac{1}{5}$$

$$= 1 \times \frac{1}{5}$$

$= \frac{1}{5}$ Is $\frac{1}{5}$ the simplest form of the product? Explain.

3. Consider $\frac{7}{16} \times \frac{12}{19}$. What is the simplest form of the product?

$$\frac{7}{16} \times \frac{12}{19} = \frac{7 \times 12}{16 \times 19}$$

$$= \frac{7 \times 2 \times 2 \times 3}{2 \times 2 \times 2 \times 2 \times 19}$$

What is the greatest common factor of numerator and denominator?

$$= \frac{2 \times 2}{2 \times 2} \times \frac{7 \times 3}{2 \times 2 \times 19}$$

$$= 1 \times \frac{7 \times 3}{2 \times 2 \times 19}$$

$= \frac{21}{76}$ Is $\frac{21}{76}$ in simplest form? Explain.

4. Use similar procedures to express the product of three (or more) rational numbers in simplest form.

## II. Practice

A. Without using pencil and paper, express each of the following products in simplest form.

1. $\frac{2}{3} \times \frac{3}{7}$ 2. $\frac{5}{8} \times \frac{3}{5}$ 3. $4 \times \frac{3}{4}$ 4. $\frac{12}{7} \times \frac{7}{3}$

B. Find the simplest form of each of the following products:

1. $\frac{5}{6} \times \frac{3}{10}$
2. $\frac{9}{12} \times \frac{8}{21}$
3. $\frac{9}{11} \times \frac{33}{25}$
4. $\frac{5}{8} \times \frac{12}{20}$
5. $6 \times 1\frac{3}{8}$
6. $2\frac{1}{4} \times 6\frac{1}{3}$
7. $\frac{9}{6} \times \frac{28}{6}$
8. $\frac{5}{6} \times \frac{10}{25}$
9. $2\frac{1}{4} \times 6\frac{1}{7}$
10. $5\frac{5}{8} \times 6\frac{2}{9}$

C. Express each product in simplest form:

1. $\frac{5}{8} \times \frac{21}{25} \times \frac{4}{7}$
2. $\frac{3}{4} \times \frac{1}{8} \times \frac{5}{9}$
3. $\frac{4}{7} \times 3\frac{1}{2} \times 6$
4. $\frac{4}{5} \times 4\frac{1}{2} \times 1\frac{2}{3}$
5. $2\frac{1}{2} \times 5\frac{1}{7} \times 2\frac{4}{5}$

D. A man walks at the rate of $3\frac{1}{8}$ miles per hour. How many miles does he walk in $1\frac{1}{5}$ hours?

E. A gallon of milk weighs $8\frac{3}{5}$ pounds. How much does a quart ($\frac{1}{4}$ gallon) weigh?

F. A boy found that every time his bicycle wheel made 1 turn he moved forward $6\frac{2}{3}$ feet. How far did he move if his wheel made $4\frac{1}{2}$ turns?

III. Summary

A. Which identity element can sometimes be used in expressing the product of two rational numbers in simplest form? (identity element of multiplication)

B. If the multiplicative identity can be used in the multiplication of rational numbers, would you use it before or after you compute the product of numerators and denominators?

C. What is the advantage of using the multiplicative identity before computing the product of numerators and denominators?

## Lesson 58

Topic: Division of Rational Numbers

Aim: To develop the meaning of reciprocal (multiplicative inverse)

Specific Objectives:

Meaning of reciprocal (multiplicative inverse)
Every number except zero has a reciprocal; the number one is its own reciprocal

Challenge: What are two numbers whose product is the multiplicative identity?

I. Procedure

A. Meaning of reciprocal

1. What is the multiplicative identity?

2. Refer to challenge.

   a. Have pupils list several pairs of numbers whose product is 1 and check, as, for example,

      $2 \times \frac{1}{2} = 1$

      $\frac{1}{3} \times 3 = 1$

      $4 \times \frac{1}{4} = 1$, and so on

   b. By what number would you multiply $\frac{2}{3}$ in order to get 1? $(\frac{3}{2})$

3. Tell pupils that when the product of two numbers is one, we call each number the reciprocal (or multiplicative inverse) of the other.

4. Since $\frac{2}{3} \times \frac{3}{2} = 1$ is a true statement, $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$, and $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$.

   What is the reciprocal of 12? of $\frac{1}{8}$? of $\frac{9}{10}$? of $\frac{17}{5}$?

5. Have pupils consider $2\frac{3}{4}$. How would we obtain the reciprocal of such a number?

a. Guide pupils to see that another name for $2\frac{3}{4}$ is $\frac{11}{4}$.

What is the reciprocal of $\frac{11}{4}$? $(\frac{4}{11})$

b. Have pupils give the reciprocal of each of the following:

$1\frac{2}{3}$ $3\frac{3}{8}$ $4\frac{1}{4}$ $8\frac{2}{5}$

B. Every number except zero has a reciprocal; the number one is its own reciprocal.

1. Consider $0 \times \square = 1$.

a. Can you replace the frame so that a true statement results? Explain.

b. Have pupils conclude that 0 does not have a reciprocal.

2. What number other than zero has no reciprocal?

After testing several other numbers suggested by the class, have the pupils conclude that it appears that every number except zero has a reciprocal.

3. Consider $1 \times \square = 1$.

a. What replacement makes this a true statement?

b. What is the reciprocal of 1?

c. Have pupils realize that the reciprocal of 1 is 1 itself.

II. Practice

A. In each of the following replace the frame so that a true statement results.

1. $5 \times \frac{1}{5} = \square$
2. $\frac{1}{9} \times 9 = \square$
3. $\frac{3}{4} \times \frac{4}{3} = \square$
4. $\frac{11}{6} \times \frac{6}{11} = \square$
5. $8 \times \square = 1$
6. $\square \times \frac{1}{7} = 1$
7. $\frac{5}{13} \times \square = 1$
8. $\square \times \frac{11}{3} = 1$

B. What are the reciprocals of the following numbers?

1. 10
2. $\frac{5}{12}$
3. $\frac{13}{1}$
4. 0
5. $3\frac{2}{7}$
6. $4\frac{7}{8}$
7. .9
8. 2.5

C. (Optional) Which is greater, a number greater than 1 or its reciprocal?

Which is greater, a number between 0 and 1 (exclusive) or its reciprocal?

III. Summary

A. What do we mean by the reciprocal of a number?

B. Does every number have a reciprocal? Explain.

C. What number is its own reciprocal?

D. What new vocabulary have we learned today? (reciprocal, multiplicative inverse)

Lessons 59 and 60

Topic: Division of Rational Numbers

Aim: To formulate a rule for dividing with rational numbers

Specific Objectives:

Division by one
Multiplying dividend and divisor by the same number
Using reciprocals in the division of rational numbers
Division with rational numbers is not commutative and not associative

Challenge: Mrs. Simon has 6 yards of material with which to make aprons for the school bazaar. She needs $\frac{3}{4}$ of a yard to make one apron. How many aprons will she be able to make from 6 yards?

I. Procedure

A. Division by one

1. Have pupils consider the result of dividing a number by 1.

a. $8 \div 1 = 8$ because $1 \times 8 = 8$

b. $15 \div 1 = 15$ because $1 \times 15 = 15$

c. $\frac{1}{2} \div 1 = \frac{1}{2}$ because $1 \times \frac{1}{2} = \frac{1}{2}$

d. $\frac{3}{4} \div 1 = \frac{3}{4}$ because $1 \times \frac{3}{4} = \frac{3}{4}$, and so on.

2. Elicit that the result of dividing a number by 1 is the number itself.

B. Multiplying dividend and divisor by the same number

1. Consider $8 \div 4 = \square$. What happens to the quotient if you multiply the dividend and the divisor by the same number?

a. Multiply the dividend and divisor in $8 \div 4 = \square$ by 2; by 3; by 5; by 8. What happens to the quotient?

b. Have pupils conclude that if the dividend and divisor are multiplied by the same counting number, the quotient remains the same.

2. Will the above conclusion be true if dividend and divisor are multiplied by the same rational number (except 0), instead of by the same counting number?

   a. Multiply the dividend and divisor in $8 \div 4 = \square$ by $\frac{1}{2}$; by $\frac{1}{4}$. What happens to the quotient?

   b. Multiply the dividend and divisor in $36 \div 12 = \square$ by $\frac{1}{3}$; by $\frac{1}{6}$. What happens to the quotient?

   c. Have pupils conclude that if a dividend and divisor are multiplied by the same rational number, the quotient remains the same.

3. Consider $48 \div 4 = \square$.

   a. If we wish to change the divisor to 1 without changing the quotient, by what rational number should we multiply dividend and divisor?

   b. Elicit that multiplying dividend and divisor of $48 \div 4 = \square$ by $\frac{1}{4}$ will change the divisor to 1 and leave the quotient unchanged.

      1) $48 \div 4 = 12$

      2) $(48 \times \frac{1}{4}) \div (4 \times \frac{1}{4}) = (48 \times \frac{1}{4}) \div 1 = 12 \div 1 = 12$

   c. How are 4 and $\frac{1}{4}$ related? (reciprocals)

4. In each of the following multiply the dividend and divisor by the same number. Multiply by the number which will give one as the new divisor. Then find the quotient.

   a. $56 \div 8$ b. $39 \div 13$ c. $40 \div 10$ d. $72 \div 12$

C. Using reciprocals in the division of rational numbers

1. Consider $6 \div \frac{1}{2} = \square$.

   a. By what number can we multiply the dividend and divisor in $6 \div \frac{1}{2} = \square$ to get one as the divisor? (2) Will the quotient be changed by this multiplication?

b. How are $\frac{1}{2}$ and 2 related?

c. Guide pupils to compute the quotient as follows:

$$6 \div \frac{1}{2} = (6 \times 2) \div (\frac{1}{2} \times 2)$$
$$= (6 \times 2) \div 1$$
$$= 12 \div 1$$
$$= 12$$

2. Refer to the challenge problem. Elicit that to solve this problem, it is necessary to compute the quotient $6 \div \frac{3}{4}$.

a. $6 \div \frac{3}{4} = (6 \times \frac{4}{3}) \div (\frac{3}{4} \times \frac{4}{3})$

Why did we choose to multiply the dividend and divisor by $\frac{4}{3}$ ?

$$= (6 \times \frac{4}{3}) \div 1$$
$$= \frac{24}{3} \div 1$$
$$= 8$$

b. Mrs. Simon can make 8 aprons.

3. In each of the following choose the number by which you can multiply the dividend and divisor to get one as the divisor. Then find the quotient.

a. $8 \div \frac{4}{5}$ b. $3 \div \frac{2}{3}$ c. $9 \div \frac{3}{4}$ d. $15 \div \frac{3}{5}$

4. Consider $\frac{1}{4} \div 3 = \square$

a. By what number will you multiply the dividend and the divisor to make the divisor one? ($\frac{1}{3}$)

b. How are 3 and $\frac{1}{3}$ related?

c. Compute the quotient.

$$\frac{1}{4} \div 3 = (\frac{1}{4} \times \frac{1}{3}) \div (3 \times \frac{1}{3})$$
$$= (\frac{1}{4} \times \frac{1}{3}) \div 1$$
$$= \frac{1}{12} \div 1 \quad = \frac{1}{12}$$

5. Consider $8 \div 1\frac{1}{3}$ .

   a. How may $1\frac{1}{3}$ be renamed?

   b. Then $8 \div 1\frac{1}{3} = 8 \div \frac{4}{3}$

   c. Complete the computation of the quotient.

6. Consider $\frac{3}{4} \div \frac{2}{3}$

   a. By what number can we multiply the dividend and the divisor to get one as the divisor? $(\frac{3}{2})$

   b. How are $\frac{2}{3}$ and $\frac{3}{2}$ related?

   c. Compute the quotient.

$$\begin{aligned} \frac{3}{4} \div \frac{2}{3} &= (\frac{3}{4} \times \frac{3}{2}) \div (\frac{2}{3} \times \frac{3}{2}) \\ &= (\frac{3}{4} \times \frac{3}{2}) \div 1 \\ &= \frac{9}{8} \div 1 \\ &= \frac{9}{8} \text{ or } 1\frac{1}{8} \end{aligned}$$

7. Have pupils conclude that to divide by a rational number, we multiply by its reciprocal.

8. Have pupils practice applying the rule for division of rational numbers.

   a. $8 \div \frac{1}{2}$

   b. $10 \div \frac{1}{5}$

   c. $\frac{2}{3} \div 4$

   d. $\frac{3}{4} \div 8$

   e. $27 \div \frac{3}{5}$

   f. $\frac{2}{5} \div \frac{2}{3}$

   g. $\frac{4}{9} \div \frac{1}{3}$

   h. $14 \div 3\frac{1}{3}$

   i. $1\frac{2}{3} \div 15$

D. Division with rational numbers is not commutative and not associative

1. Does $1\frac{1}{2} \div 1\frac{2}{3} = 1\frac{2}{3} \div 1\frac{1}{2}$ ?

   a. $1\frac{1}{2} \div 1\frac{2}{3} = \frac{3}{2} \div \frac{5}{3} = \frac{3}{2} \times \frac{3}{5} = \frac{9}{10}$

   b. $1\frac{2}{3} \div 1\frac{1}{2} = \frac{5}{3} \div \frac{3}{2} = \frac{5}{3} \times \frac{2}{3} = \frac{10}{9}$

   c. Then $1\frac{1}{2} \div 1\frac{2}{3} \neq 1\frac{2}{3} \div 1\frac{1}{2}$ . We have found a counter-example.

   Therefore, division with rational numbers is not commutative.

2. Does $(\frac{1}{2} \div \frac{1}{4}) \div \frac{1}{3} = \frac{1}{2} \div (\frac{1}{4} \div \frac{1}{3})$?

   a. $(\frac{1}{2} \div \frac{1}{4}) \div \frac{1}{3} = (\frac{1}{2} \times \frac{4}{1}) \div \frac{1}{3} = 2 \div \frac{1}{3} = 2 \times 3 = 6$

   b. $\frac{1}{2} \div (\frac{1}{4} \div \frac{1}{3}) = \frac{1}{2} \div (\frac{1}{4} \times \frac{3}{1}) = \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$

   c. Then $(\frac{1}{2} \div \frac{1}{4}) \div \frac{1}{3} \neq \frac{1}{2} \div (\frac{1}{4} \div \frac{1}{3})$. We have found a counter-example.

   Therefore , division with rational numbers is not associative.

II. Practice

A. In each of the following, replace the frame with a numeral so that a true statement results.

1. $15 \div 1 = \square$
2. $15 \div \square = 15$
3. $\frac{2}{3} \div 1 = \square$
4. $\frac{2}{3} \div \square = \frac{2}{3}$
5. $5\frac{1}{4} \div 1 = \square$
6. $5\frac{1}{4} \div \square = 5\frac{1}{4}$

B. For each of the following choose the number by which you can multiply the dividend and divisor to get one as the divisor. Then compute the quotient.

1. $6 \div \frac{1}{6}$
2. $4 \div \frac{1}{2}$
3. $8 \div \frac{1}{9}$
4. $\frac{1}{3} \div 5$
5. $\frac{5}{4} \div 12$
6. $\frac{7}{8} \div 14$

C. Compute the quotient in simplest form for each of the following.

1. $\frac{2}{3} \div \frac{1}{3}$
2. $5 \div \frac{3}{5}$
3. $\frac{7}{8} \div 4$
4. $1\frac{1}{4} \div 3$
5. $\frac{10}{3} \div \frac{1}{6}$
6. $\frac{3}{9} \div \frac{8}{3}$
7. $3\frac{1}{4} \div \frac{3}{4}$
8. $6\frac{3}{8} \div 4\frac{1}{2}$

D. A food company puts $\frac{1}{4}$ of a quart of pears into each jar. How many jars are needed for 18 quarts of pears?

E. Ann's mother bought 12 yards of linen to make dish towels. If each towel is to be $\frac{2}{3}$ of a yard long, how many towels will she be able to make?

F. How long would it take to get $14\frac{1}{2}$ gallons out of a water tank, if it is taken out at the rate of $\frac{1}{4}$ gallon per minute?

G. Sam's father drove 190 miles in $4\frac{3}{4}$ hours. What was his average speed?

III. Summary

A. We say that the number one is the identity element of division when one is the divisor. What does this mean?

B. In a division example, if the dividend and divisor are multiplied by the same rational number, what happens to the quotient?

C. In a division example, if we wish to change the divisor to 1, without changing the quotient, by what rational number should we multiply dividend and divisor?

D. What rule did we formulate for dividing rational numbers?

E. Is division of rational numbers commutative? associative? Explain.